M335 MT 1-4

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

STUDIES IN PURE MATHEMATICS

METRIC AND TOPOLOGICAL SPACES UNITS 1-4

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics
Prepared by the Course Team

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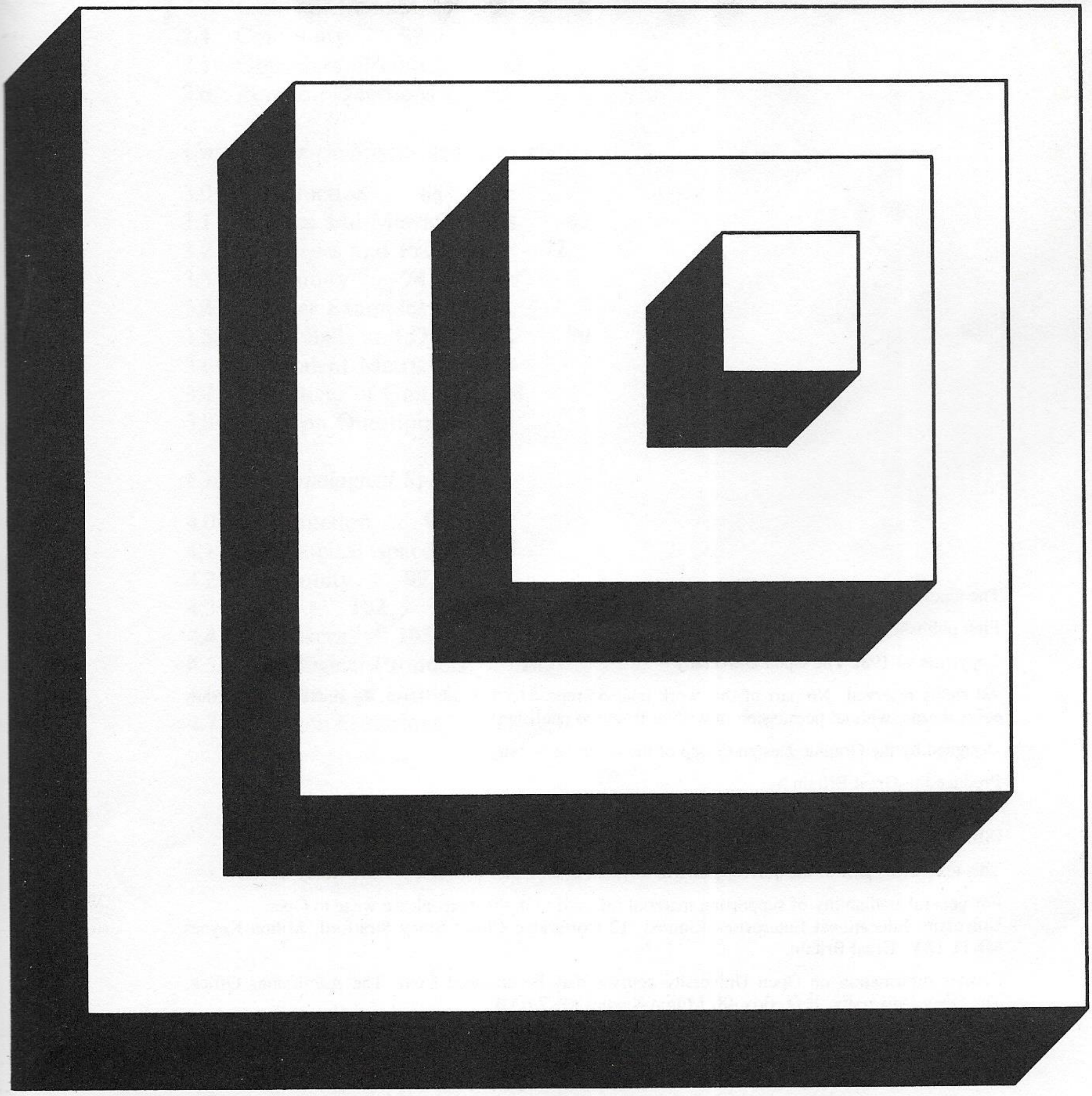

Course Team

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Introduction to the Option

The Overall Course

This course is one of the options for the Open University half-credit course M335: *Studies in Pure Mathematics.* For information about the overall course, including the continuous assessment and the final examination, see *Guide to the Course M335* and current Stop Press items.

Prerequisites

In order to study this Metric and Topological Spaces option successfully, and to benefit fully from it, it is almost essential that you have previously studied *some* course in Analysis, such as

M231 *Analysis*

or M203 *Introduction to Pure Mathematics.*

If you wish to refresh your memory of Analysis, it is better to do so *before* starting the option. If, during your study of the option, you run into any difficulty with a concept or technique from Analysis, you should then refer back to your previous course or to the relevant section (in Chapters 1–8) of

Calculus, by M. Spivak (Addison-Wesley,
World Student Series Edition, 1973).

The first two units of the option, on *set theory* and *continuity*, revise the parts of those subjects that are essential for the rest of the option; they are intended as a bridge between your previous mathematics work and the last six units, the study of *metric spaces* and *general topological spaces.*

The Set Textbook

The set textbook for this option is

Introduction to Metric and Topological Spaces, by W. A. Sutherland
(Oxford University Press, 1975).

It is essential that you possess a copy of this book.

Although you will be reading through most of the book, some sections of it have been omitted in order to cover only an acceptable total amount of material for an option of one-quarter credit value. If you have time during the course, you might well wish to read quickly through the omitted sections to see what they are about; and after the course is over we hope that you will want to do just that!

The Course Units

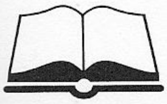
The overall material has been divided into eight more or less equal parts, which we present as 'units'. Each unit, including the associated assessment materials, is intended to involve you in about 12 hours work. The units should be studied in the order in which they appear.

Our units consist generally of guided tours through the relevant parts of the set textbook, which we refer to as S (for Sutherland). The reading passages are clearly indicated; for instance:

READ S from page 54, line 11, to the end of Section 3.2.

Most reading passages are accompanied by some notes in the unit text, the notes being referenced by a line in S. (Line $-n$ denotes the nth line up from the bottom of the page; the end of a notes section is marked ▲.)

In addition, the units contain worked Examples, Self-assessment Questions (SAQs), lists of Objectives, and Revision SAQs. The Examples and SAQs are numbered in order, starting at 1 in each unit; the end of an Example is indicated by the symbol □, where necessary, and the end of the Solution to an SAQ by the symbol ■.



From time to time, units contain additional teaching material, often in a sequence of numbered Propositions. You should regard such material as an integral part of what you should know, having the same status as material in S that you have been asked to read; it is indicated in the unit by the symbol shown in the margin.

We suggest that, in each section of a unit, you begin by marking in S the beginning and the end of the reading passage, and put a mark against each line of S on which there is a Note in the unit. Then read the brief introduction (if any), followed by the relevant parts of S. In the same working session, you should read the remaining Propositions or Examples in the unit, and tackle *all* the SAQs in the section. Then check the list of Objectives to remind you what were the key ideas that you met.

Do not feel bad if you cannot solve all the SAQs at once! However, since they are used by the Course Team as part of the teaching element of the option (and so their results are often used later on in the option), as well as for your own assessment of your understanding of the material in the section, do not omit any of them (unless they are asterisked—see below).

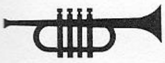
After working through the unit, have a go at the Revision SAQs. Alternatively, you may prefer to keep them for revision practice later. Revision SAQs are for self-assessment only and contain no new teaching material.

You will meet another four symbols in the left-hand margin, intended to guide you to the status of notes, examples, or SAQs.

* denotes an SAQ or portion of an SAQ that you can regard as a worked Example rather than an SAQ if you are short of time, by reading its solution rather than tackling it yourself. These questions are often further practice in a technique that you have already met; many of them, however, contain results to which we shall need to refer later on in the course, and so you should *not* just ignore such SAQs!

is a *danger* sign. It indicates an Example or an SAQ showing that some particular hypothesis in the statement of a result cannot be omitted without the result becoming (in general) false; such an example is often called a *counter-example*. Alternatively, the danger sign is used to show the falsity of some very plausible statement, or some very common pitfall on which most students stumble when first they meet a particular topic or concept.

is a *baggage* sign. It indicates either a commonly-used fact or a standard example. Such items should become part of your tool-kit as a practising mathematician. Often, they will be used several times in the option. You should remember these examples, and be able to explain why they have the properties they do.



is a *trumpet* sign. This is used to indicate some point that the writers would wish to *shout from the rooftops*, so that you never forget it. It does not have the precise meaning that the previous three symbols have; but it indicates a point in a lecture that we would preface by: 'Now, never forget that ...'!

In the list of Objectives in each unit we have tried to pick out what we feel are the *principal* concepts and results in the unit, in the sense that if you really understand *these* then you are well on your way to being on top of that unit in the course. When we say you should 'know the definition of X', for example, we mean that you should know the formal definition of X, be able to recognize when a particular example satisfies the definition and when it does not, and be able to supply an example (or any *standard* examples in the text) of it. So 'know' does not imply 'parrot fashion' knowledge, but a good grasp of what is involved and some manipulative facility with it.

An old (and wise) mathematician once said that 'one picture is worth a thousand words'. Even a crude diagram often *illuminates* a lengthy or complicated mathematical argument that you are reading, or *suggests* to you how you might attempt to tackle an SAQ on your own. (Our discussion of Venn diagrams in *Unit 1* is a good example of this.) However, you must guard against the elementary fallacy of 'proving a result *by diagram*'. A diagram *can* suggest how a careful, rigorous argument might be constructed, but a diagram *can* mislead you easily—for example, your diagram may represent only one of several possibilities that may occur (so that the 'proof' you construct will not necessarily cover all the cases that actually occur). As you read the units, notice how we use the diagrams to illustrate the written word and *not* to replace it. We very strongly encourage you as you read to draw even more diagrams for yourself; *we* do for ourselves!

In several places in the option, we need some basic facts about the positive integers. Given positive integers m and n, m is said to be a *factor* (or *divisor*) of n if the quotient $\frac{n}{m}$ is itself an integer. A positive integer p is said to be *prime* if its only factors are 1 and p. The so-called *Prime Factorization Theorem* (or Fundamental Theorem of Arithmetic) says that every positive integer can be written as a product of primes, and this factorization is unique apart from the order of writing down the prime factors. Finally, it is not hard to prove that there are infinitely many prime numbers.

Tape Sections

A number of the units contain tape sections; for these you will need access to a cassette player. In working on these sections you will, as always, need pencil and paper handy, and a pause or stop button within easy reach.

The symbol shown in the left-hand margin will indicate that there is material on one of the cassette tapes relevant to that particular section of the unit.

In addition, one cassette contains a historical discussion of how the whole subject of *topology* arose. While this material will not be assessed, we feel that it is especially valuable. Consequently, we recommend that you listen to this historical introduction several times—before you start work on the option, after you have read to the end of *Unit 4*, and after you have finished the option. Each time, you will notice something you had not noticed before, and will look at something you noticed before in a new light.

Further Reading

We hope that you will enjoy this option. If you wish to continue further your study of topology itself, the following books are good places to start.

> *Topology*, by J. Dugundji (Allyn and Bacon, 1966);
> *Counter-examples in Topology*, by L. A. Steen and J. A. Seebach Jr (Holt, Rinehart and Winston, 1970).

On the other hand, if you would like a book which contains (*inter alia*) material similar to that of this option with very many worked examples, you might consult

> *General Topology*, by S. Lipschutz (Schaum Outline Series, 1965).

UNIT 1

SETS AND FUNCTIONS

1.0 Introduction

We are going to study metric and topological spaces and those mappings between such objects which are called continuous functions. Both metric spaces and topological spaces are sets with some additional structure. Hence we begin with a survey of the ideas about sets and functions which will be needed later. Some of the ideas will be familiar and these are treated fairly briefly, but the new concepts which are introduced are discussed in more detail. We have tried in this unit, either in the text or in the SAQs, to expose you to as many as possible of the purely set-theoretical arguments which will occur later.

This unit is intended as a 'service' unit for the rest of the option. We hope that you will find it useful for reference when the new concepts are employed subsequently.

1.1 Sets

A typical dictionary definition of a set would be 'a well-defined collection of objects'. Whilst this would not serve as a mathematical definition, it is a suitable guide to mathematical usage. To specify a set we must say precisely what are its members. If A is a set then '$x \in A$' means 'x is a member of the set A'. Also, we write '$x \notin A$' to indicate that x is not a member of the set A. The members of a set are sometimes called its *elements*. Thus '$x \in A$' can also be read as 'x is an element of the set A'.

A finite set is most easily specified by listing its members within braces. For example,

$$A = \{2, 3, 5\}$$

is the set with three members, 2, 3 and 5. For infinite sets it is not, of course, possible to list the members, but we often use variants of the brace notation. For example,

$$\mathbf{R} = \{x : x \text{ is a real number}\} \qquad \text{(read ':' as 'such that')}$$

or

$$\mathbf{N} = \{1, 2, 3, \ldots\}.$$

In writing **N** in this way, we are anticipating that you will appreciate that the listing is to be continued 'in the same way'. As a second example, if we write

$$A = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\},$$

it is to be understood that A is the set whose elements are the numbers $\dfrac{1}{n}$, where $n \in \mathbf{N}$.

[We use the terms *finite set* and *infinite set* in their obvious sense; namely, a finite set is one which contains a finite number of elements, and any set which is not finite is called infinite.]

At this point, let us recall some standard notation which will be used subsequently and has already made a partial appearance above. We shall use **N**, **Z**, **Q**, **R** and **C** to denote the sets of positive integers, integers, rational numbers, real numbers and complex numbers, respectively. Also we put

$$\mathbf{R}_+ = \{t \in \mathbf{R} : t > 0\} \quad \text{and} \quad \mathbf{R}_- = \{t \in \mathbf{R} : t < 0\}.$$

Here we have also used a convenient piece of notation: in general, if A is a set and P is some condition which may or may not hold for each element of A then

$$\{x \in A : x \text{ satisfies } P\}$$

is the set whose elements are all those elements of A which satisfy the condition P. Thus, for example, $\mathbf{R}_+$ is the set of real numbers t such that $t > 0$.

The structure of the set **R** of real numbers will be discussed in Section 2.1 of *Unit 2*. We assume that you are familiar with the representation of **R** as the number line in which each real number corresponds to a point of the line.

Also we assume that you are accustomed to representing, by means of Cartesian coordinates, the points of a plane by ordered pairs (x_1, x_2) where x_1 and x_2 are real numbers. The set which consists of all such ordered pairs (x_1, x_2) is denoted by $\mathbf{R}^2$.

Two sets are *equal* if and only if they have exactly the same members. We write $A = B$ if the sets A and B are equal.

EXAMPLE 1

$$\{2,3,5\} = \{5,2,3\}$$

since these sets have the same elements; the order of listing of the elements is of no consequence.

EXAMPLE 2 Let A denote the set whose elements are the letters of the English alphabet and let

$$B = \{x \in A : x \text{ occurs in the word 'partner'}\}$$

and

$$C = \{x \in A : x \text{ occurs in the word 'pattern'}\}.$$

Then

$$B = C = \{a,\ e,\ n,\ p,\ r,\ t\}. \quad \square$$

If every member of a set B is also a member of a set A, then we say that B is a *subset* of A and write $B \subset A$. [We shall use $\subset$, rather than $\subseteq$ which you might have met before, to conform with the usage in S.] Of course, if $C \subset B$ and $B \subset A$ then $C \subset A$. It should be noted that $A \subset A$, since every member of the set A is a member of A. A key fact is that if A and B are sets then $A = B$ if and only if $A \subset B$ and $B \subset A$. Thus, *the equality of two sets is established by showing that each is a subset of the other.*

If $B \subset A$ and $B \neq A$ then B is called a *proper subset* of A. To show that a set B is *not* a subset of a set A, one must find an element x of B such that $x \notin A$. The empty set, denoted by $\varnothing$, is the unique set which has no elements. Thus $\varnothing \subset A$ for every set A.

We have not placed any restriction on the nature of the elements of a set. In our work in topology we shall frequently consider sets whose elements are themselves sets. We consider now an important instance. The *power set* of a set A, denoted by 2^A, is the set whose elements are the subsets of the given set A. Thus

$$2^A = \{B : B \subset A\}.$$

Note that $\varnothing \in 2^A$ and $A \in 2^A$.

EXAMPLE 3 If $A = \{a, b\}$ then

$$2^A = \{\varnothing, \{a\}, \{b\}, A\}. \quad \square$$

Some evidence for the appropriateness of denoting the power set of A by 2^A is supplied by the fact that if A is a set with n elements then 2^A has 2^n elements (SAQ 3).

Note that if A and B are sets with $B \subset A$ then $2^B \subset 2^A$. For if $C \in 2^B$ then $C \subset B$ so that $C \subset A$ and hence $C \in 2^A$.

Next we refresh your memory about two operations on sets. If A and B are sets, their *union*, denoted by $A \cup B$, is the set whose elements are all the elements of A together with all the elements of B. Thus

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The *intersection* of A and B, denoted by $A \cap B$, is the set whose elements are all those elements common to both A and B. Thus

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Of course, it may be the case that A and B have no common elements so that $A \cap B = \varnothing$; such sets are said to be *disjoint*.

For future use, we note some (fairly obvious) facts about any two sets A and B. The following statements are *equivalent* (that is, the truth of *any one* of them implies the truth of the other two):

(i) $A \cup B = A$; (ii) $A \cap B = B$; (iii) $B \subset A$.

The following statements are also equivalent (where $\Rightarrow$ means implies):

(i) $A \cap B = \varnothing$; (ii) $x \in A \Rightarrow x \notin B$; (iii) $x \in B \Rightarrow x \notin A$.

If $A_1, \ldots, A_n$ are sets, then $A_1 \cup A_2 \cup \ldots \cup A_n$, also denoted by $\bigcup_{i=1}^{n} A_i$, is the set which consists of those elements which belong to at least one of the sets A_i; and $A_1 \cap A_2 \cap \cdots \cap A_n$, also denoted by $\bigcap_{i=1}^{n} A_i$, is the set which consists of those elements which belong to every one of the sets A_i. Thus

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for } \textit{some } i \text{ with } 1 \le i \le n\}$$

and

$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for } \textit{every } i \text{ with } 1 \le i \le n\}.$$

In the next section we shall consider an extension of the set-theoretic operations of union and intersection.

Now we introduce another operation on sets which we shall use often in later work. If A and B are sets, then the *relative complement* of B in A, denoted by $A - B$, is the subset of A consisting of those elements which are not members of B. Thus

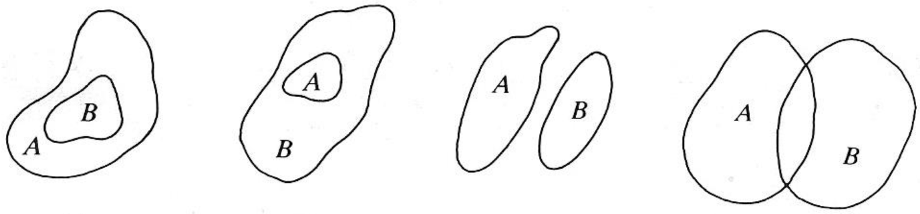
$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

If $B \subset A$, then $A - B$ is called the *complement* of B in A.

EXAMPLE 4 If $X = \{a, b, c\}$, $A = \{a, b\}$ and $B = \{b, c\}$ then

$$A - B = X - B = \{a\} \quad \text{and} \quad B - A = X - A = \{c\}. \quad \square$$

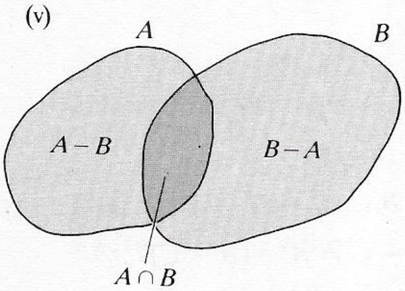
A useful means of visualizing relations between sets is provided by *Venn diagrams*. We represent each set by a region in the plane. The relationship $B \subset A$ can be represented by a diagram of form (i) and $A \subset B$ by a diagram of form (ii), whilst if A and B are disjoint we should draw a diagram of form (iii). If neither of these relationships hold, we should draw a diagram of form (iv).

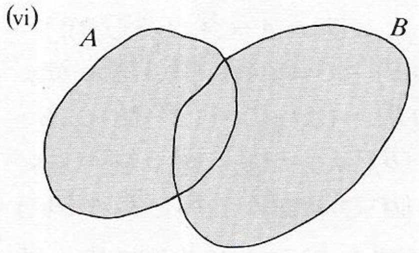


(i) (ii) (iii) (iv)

Notice that diagram (iv) is a more general diagrammatic representation of the possible relationships between A and B than the others, in the sense that (i), (ii) and (iii) are special cases of (iv) when one of the three regions in (iv) is actually the empty set.

$A \cap B$, $A - B$ and $B - A$ corresponding to sets A, B as in diagram (iv) are represented by shaded areas as in diagram (v), and in diagram (vi) we have shaded the area representing $A \cup B$.





Venn diagrams *cannot be used for the proofs* of set-theoretic formulae but they are very useful for checking that results are plausible and for motivating and keeping track of formal arguments. We hope that you will *always* have pencil and paper to hand and draw pictures as an aid to thought in all your work for this option.

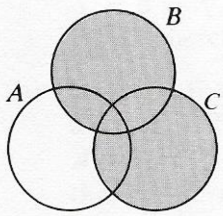
EXAMPLE 5 We shall later on need the identities:

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; and

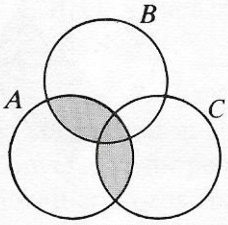
(ii) $A - (B \cup C) = (A - B) \cap (A - C)$,
$A - (B \cap C) = (A - B) \cup (A - C)$.

We shall establish identities containing these as special cases in the next section. So here we shall prove only the first of the four identities, and invite you (if you have time!) to follow the same procedure to prove the other three.

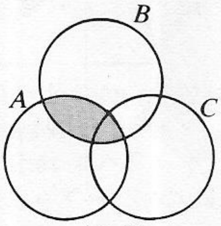
After considering the following four Venn diagrams, you will believe that the first identity is valid.



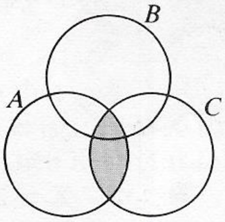
$B \cup C$



$A \cap (B \cup C)$



$A \cap B$



$A \cap C$

Now let us give a formal proof of the identity. We use the symbol $\Longleftrightarrow$ as an abbreviation for 'if and only if' or 'implies and is implied by'.

$$\begin{aligned} x \in A \cap (B \cup C) &\Longleftrightarrow x \in A \text{ and } x \in B \cup C \\ &\Longleftrightarrow x \in A \textit{ and } (x \in B \text{ or } x \in C) \\ &\Longleftrightarrow (x \in A \text{ and } x \in B) \textit{ or } (x \in A \text{ and } x \in C) \\ &\Longleftrightarrow (x \in A \cap B) \textit{ or } (x \in A \cap C) \\ &\Longleftrightarrow x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

SAQ 1 Let

$$A = \{x \in \mathbf{R} : x^2 = 1\} \quad \text{and} \quad B = \{x \in \mathbf{R} : x^4 = 1\}.$$

Prove that $A = B$. Prove that if

$$A' = \{z \in \mathbf{C} : z^2 = 1\} \quad \text{and} \quad B' = \{z \in \mathbf{C} : z^4 = 1\}$$

then A' is a proper subset of B'.

SAQ 2 Let $A = \{(x_1, x_2) \in \mathbf{R}^2 : x_2 \geq x_1\}$ and $B = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 \leq 1\}$.

Sketch the sets A, B, $A \cap B$, $A \cup B$, $A - B$ and $B - A$.

SAQ 3 (a) If $A = \{a, b, c\}$, find 2^A.

(b) Show that, if B is a set with n (≥ 1) elements, then 2^B has 2^n elements.

SAQ 4 Let A, B and C be sets. Prove that

*(a) $A \cup (A \cap B) = A = A \cap (A \cup B)$;

(b) $(A \cup B) - C = (A - C) \cup (B - C)$;

*(c) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$;

*(d) $A - (B - C) = (A - B) \cup (A \cap C)$;

*(e) $A - (B \cup C) = (A - B) - C = (A - C) - B$.

SAQ 5 Let A, B and C be sets. Is it true that $(A - B) - C = A - (B - C)$?

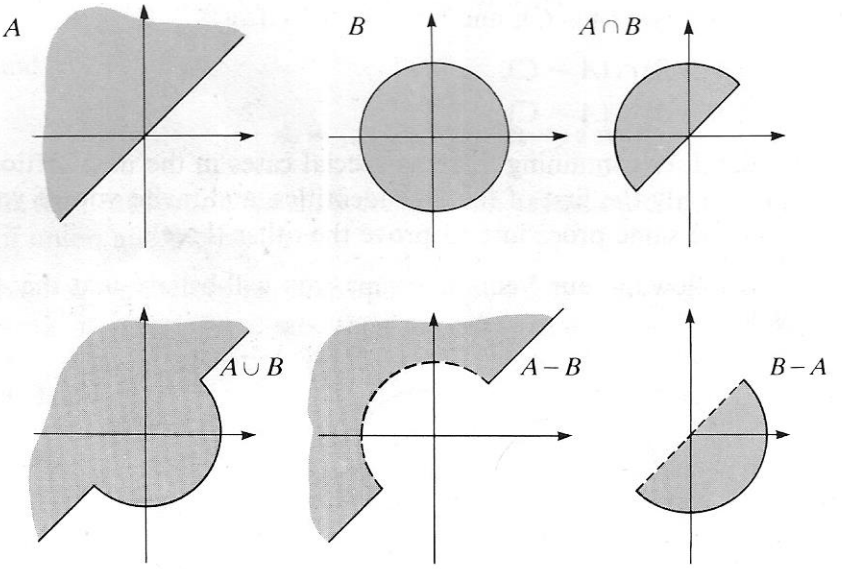
SAQ 6 Let A and B be subsets of a set X. Show that

(a) $A \subset B \Longleftrightarrow X - B \subset X - A$; and

(b) $A \subset X - B \Longleftrightarrow A \cap B = \varnothing$.

SOLUTION TO SAQ 1 If $x \in \mathbf{R}$, then $x^2 = 1$ if and only if $x = +1$ or $x = -1$. Thus $A = \{+1, -1\}$. Also if $x \in \mathbf{R}$, then $x^4 = 1$ if and only if $x^2 = 1$ (since $x^2 \neq -1$ if $x \in \mathbf{R}$). Thus $B = \{+1, -1\} = A$. Finally, $A' = \{+1, -1\}$, but $B' = \{+1, -1, +i, -i\}$, so that A' is a proper subset of B'. ■

SOLUTION TO SAQ 2



■

Remark. Throughout this option we shall use the (normal) convention of indicating those portions of the boundary not included in the set by a broken line, and those portions which are included by a continuous line. Similarly, a particular point that is included is indicated by a black dot ●, and one not included by a hollow dot ○.

SOLUTION TO SAQ 3 (a) $2^A = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$

(b) If B is a set with one element then $2^B = \{\varnothing, B\}$, so the asserted result is true if $n = 1$. Suppose that the result is true for some $n \geq 1$ and let $B = \{b_1, \ldots, b_n, b_{n+1}\}$. Put $B' = \{b_1, \ldots, b_n\}$. If $C \subset B$ then either $C \subset B'$ or $C = C' \cup \{b_{n+1}\}$ where $C' \subset B'$. By hypothesis, $2^{B'}$ has 2^n elements, so that 2^B has $2^n + 2^n = 2^{n+1}$ elements. The result is thus established by mathematical induction. ■

SOLUTION TO SAQ 4 (a) Since $A \cap B \subset A$ we have $A \cup (A \cap B) = A$, and since $A \subset A \cup B$ we have $A \cap (A \cup B) = A$.

(b)
$$\begin{aligned} x \in (A \cup B) - C &\Longleftrightarrow (x \in A \text{ or } x \in B) \text{ and } x \notin C \\ &\Longleftrightarrow x \in A - C \text{ or } x \in B - C \\ &\Longleftrightarrow x \in (A - C) \cup (B - C). \end{aligned}$$

(c) First we note that

$$\begin{aligned} x \in A - (A \cap B) &\iff x \in A \text{ and } x \notin A \cap B \\ &\iff x \in A \text{ and } x \notin B \\ &\iff x \in A - B. \end{aligned}$$

Thus $A - (A \cap B) = A - B$ and similarly $B - (A \cap B) = B - A$. It now follows from (b) that

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

(d) First we note that $x \in A - (B - C) \iff x \in A$ and $x \notin B - C$. But $x \notin B - C \iff x \notin B$ or $x \in C$. Thus

$$\begin{aligned} x \in A - (B - C) &\iff (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \in C) \\ &\iff x \in A - B \text{ or } x \in A \cap C \\ &\iff x \in (A - B) \cup (A \cap C). \end{aligned}$$

(e) $$\begin{aligned} x \in A - (B \cup C) &\iff x \in A \text{ and } x \notin B \cup C \\ &\iff x \in A \text{ and } x \notin B \text{ and } x \notin C \\ &\iff x \in A - B \text{ and } x \notin C \\ &\iff x \in (A - B) - C. \end{aligned}$$

Thus $A - (B \cup C) = (A - B) - C$. By interchanging the roles of B and C we see that

$$A - (B \cup C) = A - (C \cup B) = (A - C) - B. \quad \blacksquare$$

SOLUTION TO SAQ 5 Consideration of a Venn diagram will have led you to reject this formula as improbable. It remains to supply a specific counter-example. Let $A = \{a, b, c\}$, $B = \{b\}$ and $C = \{c\}$, where a, b, c are distinct. Then $A - B = \{a, c\}$ so that $(A - B) - C = \{a\}$. But $B - C = \{b\}$ so that $A - (B - C) = \{a, c\}$. $\blacksquare$

SOLUTION TO SAQ 6 (a) Suppose that $A \subset B$. If $x \in X - B$ then $x \notin B$, so that $x \notin A$ and hence $x \in X - A$. Therefore $X - B \subset X - A$. Conversely, suppose that $X - B \subset X - A$. If $x \in A$ then $x \notin X - A$, so that $x \notin X - B$ and hence $x \in B$. Therefore $B \subset A$.

(b) Suppose that $A \subset X - B$. If $x \in A \cap B$ then $x \in A$ so that $x \in X - B$; and also $x \in B$. This contradiction shows that no such x exists; that is, that $A \cap B = \varnothing$. Conversely, suppose that $A \cap B = \varnothing$. Then if $x \in A$ it follows that $x \notin B$, or in other words that $x \in X - B$; thus $A \subset X - B$. $\blacksquare$

1.2 Indexed Families of Sets

The operations of union and intersection have been defined for finite collections of sets. For our work in topology we need to extend the operations to infinite collections of sets. To handle these infinite operations most conveniently we introduce the idea of an indexed family of sets.

Let I be a non-empty set. An *indexed family of sets* is determined by associating a set A_i with each element i of I. The indexed family of sets, in which the set A_i is associated with the element i of I, is denoted by $(A_i)_{i \in I}$. The set I is called the *indexing set* of the indexed family $(A_i)_{i \in I}$ and the elements of I are called *indices*. We note that it is *not* assumed that sets in an indexed family with distinct indices are different.

An indexed family of sets with indexing set $\mathbf{N}$ is determined by specifying a set A_n for each positive integer n.

EXAMPLE 6 For each n in $\mathbf{N}$ let

$$A_n = \{x \in \mathbf{Z}\text{: there exists } y \text{ in } \mathbf{Z} \text{ such that } x = ny\}.$$

Thus $A_1 = \mathbf{Z}$,

$A_2 = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$,

$A_3 = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$, and so on.

We thus obtain an indexed family $(A_n)_{n \in \mathbf{N}}$ of sets.

EXAMPLE 7 Let P be the set of prime numbers 2, 3, 5, ... We might consider the family $(B_p)_{p \in P}$ of subsets of $\mathbf{Z}$ where

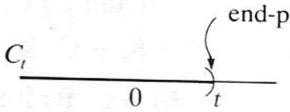
$$B_p = \{\ldots, -2p, -p, 0, p, 2p, \ldots\}.$$

Remark. Examples of indexed families with $J = \{n \in \mathbf{Z} : n \geq 0\}$ as indexing set are already very familiar to the M203 student: a sequence of nested closed intervals (M203 *Unit 8*, page 35) is an indexed family $(I_n)_{n \in J}$ where I_n is a closed interval in $\mathbf{R}$ and $I_{n+1} \subset I_n$ for each n in J.

M203

EXAMPLE 8 For each t in $\mathbf{R}_+$, let

$$C_t = \{x \in \mathbf{R} : x < t\}.$$



round bracket indicates that end-point is not in the set

Then $(C_t)_{t \in \mathbf{R}_+}$ is an indexed family of sets. ☐

The set theoretic operations of union and intersection can be extended to indexed families of sets.

DEFINITIONS Let $(A_i)_{i \in I}$ be an indexed family of sets. The *union* of the family $(A_i)_{i \in I}$ is the set

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for } \textit{some } i \text{ in } I\}.$$

The *intersection* of the family $(A_i)_{i \in I}$ is the set

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for } \textit{every } i \text{ in } I\}.$$

Occasionally when it is clear what I is, we shall abbreviate this notation to $\bigcap A_i$. ▲

The following properties of unions and intersections are often used.

PROPOSITION 1 If $(A_i)_{i \in I}$ is an indexed family of sets then

$$\bigcap_{i \in I} A_i \subset A_j \subset \bigcup_{i \in I} A_i$$

for all $j \in I$.

PROPOSITION 2 Let $(A_i)_{i \in I}$ be an indexed family of sets. If $A_i \subset B$ for every i in I then $\bigcup_{i \in I} A_i \subset B$ and if $C \subset A_i$ for every i in I then $C \subset \bigcap_{i \in I} A_i$.

In the case of an indexed family $(A_n)_{n \in \mathbf{N}}$, with indexing set $\mathbf{N}$, we usually write

$$\bigcup_{n \in \mathbf{N}} A_n = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \bigcap_{n \in \mathbf{N}} A_n = \bigcap_{n=1}^{\infty} A_n.$$

Thus $\bigcup_{n=1}^{\infty} A_n$ consists of those x such that $x \in A_n$ for *at least one* positive integer n, whilst $\bigcap_{n=1}^{\infty} A_n$ consists of those x such that $x \in A_n$ for *every* positive integer n.

Also, if $J = \{n \in \mathbf{Z} : n \geq 0\}$ and $(A_n)_{n \in J}$ is an indexed family of sets, we usually write

$$\bigcup_{n \in J} A_n = \bigcup_{n=0}^{\infty} A_n \quad \text{and} \quad \bigcap_{n \in J} A_n = \bigcap_{n=0}^{\infty} A_n.$$

M203 *Remark.* The M203 student will recognize that a sequence $(I_n)_{n \in J}$ of nested closed intervals is a nest (M203 *Unit 8*, page 36) if and only if $\bigcap_{n=0}^{\infty} I_n$ contains precisely one element.

EXAMPLE 9 Let us find the union and intersection of the family $(A_n)_{n \in \mathbf{N}}$ where

$$A_n = \{\ldots, -2n, -n, 0, n, 2n, \ldots\}$$

for each positive integer n. Since $A_n \subset A_1 = \mathbf{Z}$ for each n, it follows that $\bigcup_{n=1}^{\infty} A_n = \mathbf{Z}$. Clearly $0 \in A_n$ for each n in $\mathbf{N}$ so that $0 \in \bigcap_{n=1}^{\infty} A_n$. If $m \in \mathbf{Z}$ and $m \neq 0$ then $m \notin A_{|m|+1}$ so that there is no non-zero integer m such that $m \in A_n$ for every positive integer n. Therefore $\bigcap_{n=1}^{\infty} A_n = \{0\}$.

EXAMPLE 10 Consider the indexed family $(B_p)_{p \in P}$ of subsets of $\mathbf{Z}$, where P is the set of prime numbers and

$$B_p = \{\ldots, -2p, -p, 0, p, 2p, \ldots\}$$

for each p in P. Since every integer greater than 1 can be written as a product of prime numbers, it follows that if $n \in \mathbf{Z}$ and $|n| \geq 2$ then there exists a prime number q such that q divides $|n|$ and hence $n \in B_q$. Plainly $0 \in B_p$ for every p in P. Also there is no p in P such that $1 \in B_p$ or $-1 \in B_p$. Therefore

$$\bigcup_{p \in P} B_p = \mathbf{Z} - \{+1, -1\}.$$

If $n \in \mathbf{Z}$ and $n \neq 0$ then, since the set of prime numbers is infinite, there exists q in P such that $q > |n|$, from which it follows that $n \notin B_q$. Thus there is no non-zero integer n such that $n \in B_p$ for every p in P. Therefore $\bigcap_{p \in P} B_p = \{0\}$.

EXAMPLE 11 For the indexed family $(C_t)_{t \in \mathbf{R}_+}$, where $C_t = \{x \in \mathbf{R} : x < t\}$ for each t in $\mathbf{R}_+$, we have

$$\bigcup_{t \in \mathbf{R}_+} C_t = \mathbf{R} \quad \text{and} \quad \bigcap_{t \in \mathbf{R}_+} C_t = \{x \in \mathbf{R} : x \leq 0\}.$$

Firstly, $C_t \subset \mathbf{R}$ for each t in $\mathbf{R}_+$ so that $\bigcup_{t \in \mathbf{R}_+} C_t \subset \mathbf{R}$. But if $x \in \mathbf{R}$, then there exists t_0 in $\mathbf{R}_+$ such that $x < t_0$; $x \in C_{t_0}$ so that $x \in \bigcup_{t \in \mathbf{R}_+} C_t$. Thus $\mathbf{R} \subset \bigcup_{t \in \mathbf{R}_+} C_t$ and so $\bigcup_{t \in \mathbf{R}_+} C_t = \mathbf{R}$. If $x \in \mathbf{R}$ and $x \leq 0$ then $x < t$ if $t \in \mathbf{R}_+$, so that $x \in C_t$ for all $t \in \mathbf{R}_+$ and hence $x \in \bigcap_{t \in \mathbf{R}_+} C_t$. Therefore

$$\{x \in \mathbf{R} : x \leq 0\} \subset \bigcap_{t \in \mathbf{R}_+} C_t.$$

If $x \in \bigcap_{t \in \mathbf{R}_+} C_t$ then $x \in C_t$ for all $t > 0$, so that $x < t$ for all $t > 0$; hence $x \leq 0$. Therefore $\bigcap_{t \in \mathbf{R}_+} C_t \subset \{x \in \mathbf{R} : x \leq 0\}$, so that $\bigcap_{t \in \mathbf{R}_+} C_t = \{x \in \mathbf{R} : x \leq 0\}$. ☐

Before considering more examples we obtain an extremely important result relating the operations of union and intersection with complementation.

PROPOSITION 3 *De Morgan's Laws* If B is a set and $(A_i)_{i\in I}$ is an indexed family of sets then

(i) $B - (\bigcup_{i\in I} A_i) = \bigcap_{i\in I} (B - A_i)$

and

(ii) $B - (\bigcap_{i\in I} A_i) = \bigcup_{i\in I} (B - A_i)$.

Remark. This Proposition generalizes Example 5(ii).

PROOF (i) If $x \in B - (\bigcup_{i\in I} A_i)$, then $x \in B$ and $x \notin \bigcup_{i\in I} A_i$. Since $x \notin \bigcup_{i\in I} A_i$, there is no index i for which $x \in A_i$. Thus for every index i we have $x \notin A_i$, so that $x \in B$ and, for every index i, $x \notin A_i$. Therefore $x \in B - A_i$ for all $i \in I$, and so $x \in \bigcap_{i\in I} (B - A_i)$. This shows that

$$B - (\bigcup_{i\in I} A_i) \subset \bigcap_{i\in I} (B - A_i).$$

Conversely, suppose that $y \in \bigcap_{i\in I} (B - A_i)$. Then $y \in B - A_i$ for every i in I, so that for every i in I we have $y \in B$ and $y \notin A_i$. Since there is no index i such that $y \in A_i$ we see that $y \notin \bigcup_{i\in I} A_i$. But $y \in B$ so that $y \in B - (\bigcup_{i\in I} A_i)$. This shows that

$$\bigcap_{i\in I} (B - A_i) \subset B - (\bigcup_{i\in I} A_i),$$

and formula (i) is established. Formula (ii) is obtained in a similar way. ■

We must take care not to assume that properties of finite intersections and unions necessarily hold for the operations of infinite intersection and infinite union. The next two examples illustrate perhaps unexpected possibilities.

EXAMPLE 12 A family $(A_n)_{n\in\mathbf{N}}$ can have an empty intersection even though A_n is non-empty and $A_{n+1} \subset A_n$ for each n. For each n in $\mathbf{N}$ let

$$A_n = \{k \in \mathbf{N} : k \geq n\}.$$

Then A_n is a non-empty set and $A_{n+1} \subset A_n$ for each n. But $\bigcap_{n=1}^{\infty} A_n = \varnothing$, since $k \notin A_{k+1}$ if $k \in \mathbf{N}$, so there is no integer k such that $k \in A_n$ for every n.

EXAMPLE 13 Families $(B_n)_{n\in\mathbf{N}}$ and $(C_n)_{n\in\mathbf{N}}$, such that $B_n \subset B_{n+1}$ and $C_n \subset C_{n+1}$ for each n, can have the same union, even though B_n is a proper subset of C_n for each n. For each n in $\mathbf{N}$, let

$$B_n = \{k \in \mathbf{N} : k \leq n\} \quad \text{and} \quad C_n = \{k \in \mathbf{N} : k \leq 2n\}.$$

Then B_n is a proper subset of C_n for each n. Since $B_n \subset C_n \subset \mathbf{N}$ for every n, it follows that

$$\bigcup_{n=1}^{\infty} B_n \subset \bigcup_{n=1}^{\infty} C_n \subset \mathbf{N}.$$

But if $n \in \mathbf{N}$ then $n \in B_n$ so that $\mathbf{N} \subset \bigcup_{n=1}^{\infty} B_n$ and hence

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n = \mathbf{N}.$$

SAQ 7 Let J be an infinite subset of $\mathbf{N}$ and for each n in J let $A_n = \{n, 2n, 3n, \ldots\}$. Prove that $\bigcap_{n\in J} A_n = \varnothing$.

SAQ 8 For each positive integer n let

$$A_n = \{(x_1, x_2) \in \mathbf{R}^2 : n \leq x_1 \leq n + 1,\ n - 1 \leq x_2 \leq n\}.$$

Sketch the subset $\bigcup\limits_{n=1}^{\infty} A_n$ of $\mathbf{R}^2$.

SAQ 9 Let A be a set and let $(B_i)_{i \in I}$ be an indexed family of sets. Prove that

$$A \cap (\bigcup_{i \in I} B_i) = \bigcup_{i \in I} (A \cap B_i)$$

and

$$A \cup (\bigcap_{i \in I} B_i) = \bigcap_{i \in I} (A \cup B_i).$$

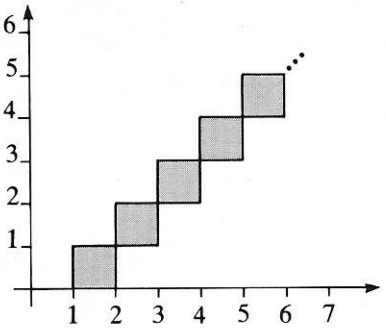
State these results for the case in which $I = \{1, 2\}$.

SAQ 10 Let A be a set and for each x in A let B_x be a subset of A such that $x \in B_x$. Prove that $\bigcup\limits_{x \in A} B_x = A$.

SOLUTION TO SAQ 7 Let $k \in \mathbf{N}$. Since J is an infinite subset of $\mathbf{N}$, there exists some m in J such that $k < m$. Then $k \notin A_m$. Hence

$$\bigcap_{n \in J} A_n = \varnothing. \quad ■$$

SOLUTION TO SAQ 8



■

SOLUTION TO SAQ 9

$$\begin{aligned} x \in A \cap (\bigcup_{i \in I} B_i) &\iff x \in A \text{ and } x \in \bigcup_{i \in I} B_i \\ &\iff x \in A \text{ and } x \in B_i \text{ for some } i \in I \\ &\iff x \in A \cap B_i \text{ for some } i \in I \\ &\iff x \in \bigcup_{i \in I} (A \cap B_i). \end{aligned}$$

$$\begin{aligned} x \in A \cup (\bigcap_{i \in I} B_i) &\iff x \in A \text{ or } x \in B_i \text{ for all } i \in I \\ &\iff x \in A \cup B_i \text{ for all } i \in I \\ &\iff x \in \bigcap_{i \in I} (A \cup B_i). \end{aligned}$$

If $I = \{1, 2\}$, we have

$$A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_2)$$

and

$$A \cup (B_1 \cap B_2) = (A \cup B_1) \cap (A \cup B_2). \quad ■$$

Remark. Observe that when $I = \{1, 2\}$ these results are just those of Example 5(i).

SOLUTION TO SAQ 10 Since $B_x \subset A$ for each x in A it follows that $\bigcup\limits_{x \in A} B_x \subset A$. But if $x_0 \in A$ then $x_0 \in B_{x_0} \subset \bigcup\limits_{x \in A} B_x$. Hence $\bigcup\limits_{x \in A} B_x = A$. ■

1.3 Functions

Let A and B be sets. You will remember that a *map* or *function*

$$f:A\longrightarrow B$$

associates with each element a of A a unique element $f(a)$ of B. The set A is called the *domain* of f and the set B is called the *codomain* of f. The element $f(a)$ of B associated with an element a of A is called the *image* of a under f. The function f from A to B which associates the element $f(a)$ of B with each element a of A is sometimes denoted by

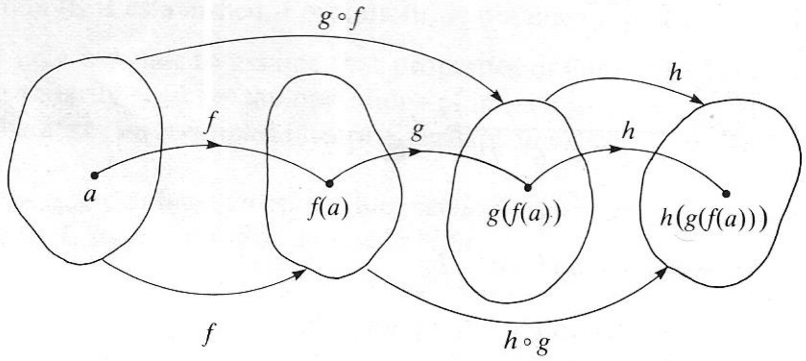
$$f:a\longmapsto f(a), \quad a\in A.$$

Two functions f and g are *equal*, and we write $f=g$, if and only if they have the same domain, the same codomain, and $f(a)=g(a)$ for each element a of their domain.

If $f:A\longrightarrow B$ and $g:B\longrightarrow C$ are functions, then for each a in A, $f(a)$ is an element of B so that $g(f(a))$ is an element of C. The *composition* of f and g is the function

$$g\circ f:a\longmapsto g(f(a)), \quad a\in A,$$

with domain A and codomain C. We note that for the composition of functions f and g to be defined, it is necessary that the codomain of f is the domain of g. The following picture will help you to verify that if $f:A\longrightarrow B$, $g:B\longrightarrow C$ and $h:C\longrightarrow D$ are functions then $h\circ(g\circ f)=(h\circ g)\circ f$.



If $f:A\longrightarrow B$ is a function and C is a subset of A then the *image of C under f* is the subset of B consisting of those elements of B which are images under f of elements of C. The image of C under f is usually denoted by $f(C)$, so that

$$f(C)=\{b\in B:b=f(a)\text{ for some }a\text{ in }C\}.$$

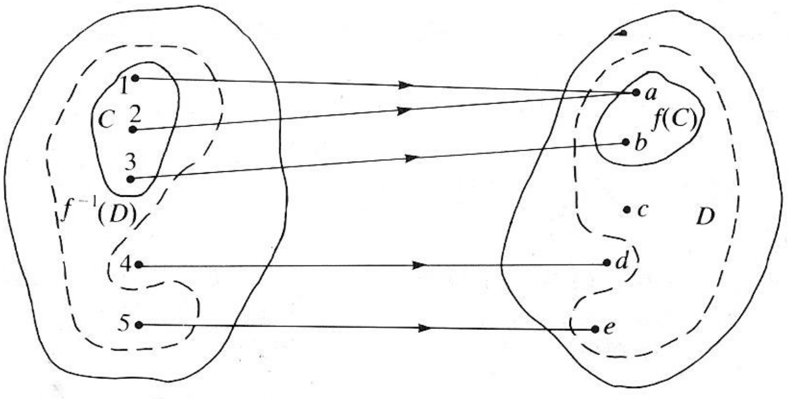
Next, if $f:A\longrightarrow B$ is a function and D is a subset of B then the *inverse image* of D under f, denoted by $f^{-1}(D)$, is the subset of A consisting of those elements of A whose images under f are elements of D. Thus

$$f^{-1}(D)=\{a\in A:f(a)\in D\}.$$

PROPOSITION 4 If $f:A\longrightarrow B$ is a function, $C\subset A$ and $D\subset B$ then $C\subset f^{-1}(D)$ if and only if $f(C)\subset D$.

PROOF If $C\subset f^{-1}(D)$ and $b\in f(C)$ then $b=f(a)$ for some $a\in C$, so that $b=f(a)$ where $a\in f^{-1}(D)$. Since $a\in f^{-1}(D)$, it follows that $b=f(a)\in D$ and hence $f(C)\subset D$. Conversely, if $f(C)\subset D$ then for each a in C we have $f(a)\in f(C)\subset D$, so that $a\in f^{-1}(D)$. Therefore $C\subset f^{-1}(D)$. ■

EXAMPLE 14 Let $A=\{1, 2, 3, 4, 5\}$, $B=\{a, b, c, d, e\}$ and let $f:A\longrightarrow B$ be the function given by $f(1)=f(2)=a$, $f(3)=b$, $f(4)=d$, $f(5)=e$.



If $C = \{1,2,3\}$ then $f(C) = \{a,b\}$. If $D = \{a,b,c,e\}$ then $f^{-1}(D) = \{1,2,3,5\}$.

SAQ 11 (a) Let $f:\mathbf{R} \longrightarrow \mathbf{R}$ be given by $f(x) = x^2, \quad x \in \mathbf{R}$.

Find $f^{-1}([4,9])$.

(b) Let $g:\mathbf{R} \longrightarrow \mathbf{R}$ be given by $g(x) = \sin x, \quad x \in \mathbf{R}$.

Find $g^{-1}([0,1])$.

SOLUTION TO SAQ 11 (a) $f^{-1}([4,9)] = \{x \in \mathbf{R}: f(x) \in [4,9]\}$

$= \{x \in \mathbf{R}: 4 \le x^2 \le 9\}$.

Since $4 \le x^2 \le 9$ if and only if either $2 \le x \le 3$ or $-3 \le x \le -2$, we see that
$$f^{-1}([4,9]) = [-3,-2] \cup [2,3].$$

(b) $g^{-1}([0,1]) = \{x \in \mathbf{R}: g(x) \in [0,1]\}$

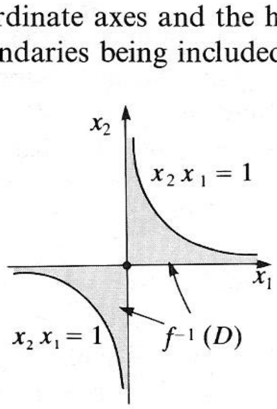
$= \{x \in \mathbf{R}: 0 \le \sin x \le 1\}$.

Now $0 \le \sin x \le 1$ if and only if $2n\pi \le x \le (2n+1)\pi$ for some integer n. Therefore
$$g^{-1}([0,1]) = \bigcup_{n \in \mathbf{Z}} [2n\pi, (2n+1)\pi]. \quad ■$$

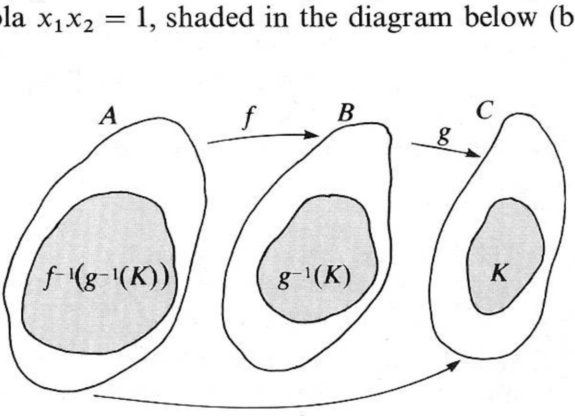
EXAMPLE 15 Consider the function $f:\mathbf{R}^2 \longrightarrow \mathbf{R}$ given by
$$f(x_1,x_2) = x_1x_2, \quad (x_1,x_2) \in \mathbf{R}^2.$$
Then
$$f^{-1}(\mathbf{R}_+) = \{(x_1,x_2) \in \mathbf{R}^2 : x_1x_2 > 0\}.$$

Since $x_1x_2 > 0$ if and only if *either* $x_1 > 0$ and $x_2 > 0$ *or* $x_1 < 0$ and $x_2 < 0$, we see that $f^{-1}(\mathbf{R}_+)$ consists of all points of the plane lying in the first or third quadrants (and not on the axes). Also, if D is the closed interval $[0,1]$ in $\mathbf{R}$, then
$$f^{-1}(D) = \{(x_1,x_2) \in \mathbf{R}^2 : 0 \le x_1x_2 \le 1\}.$$

Now $x_1x_2 = 0$ if and only if either $x_1 = 0$ or $x_2 = 0$. Also if $0 < c \le 1$ then $x_1x_2 = c$ is the equation of a rectangular hyperbola in the first and third quadrants. Thus we see that $f^{-1}(D)$ is the subset of the plane, bounded by the coordinate axes and the hyperbola $x_1x_2 = 1$, shaded in the diagram below (both boundaries being included).



Example 15



$(g \circ f)^{-1}(K) = ?$

Proposition 5

PROPOSITION 5 If $f:A \longrightarrow B$ and $g:B \longrightarrow C$ are functions and K is a subset of C then

$$(g \circ f)^{-1}(K) = f^{-1}(g^{-1}(K)).$$

PROOF Let $a \in (g \circ f)^{-1}(K)$. Then $(g \circ f)(a) \in K$ so that $f(a) \in B$ and $g(f(a)) \in K$, from which we deduce that $f(a) \in g^{-1}(K)$ and hence that $a \in f^{-1}(g^{-1}(K))$. Thus $(g \circ f)^{-1}(K) \subset f^{-1}(g^{-1}(K))$.

Conversely, if $a \in f^{-1}(g^{-1}(K))$, then $f(a) \in g^{-1}(K)$ so that $g(f(a)) \in K$. Thus $(g \circ f)(a) \in K$ and hence $a \in (g \circ f)^{-1}(K)$. Therefore $f^{-1}(g^{-1}(K)) \subset (g \circ f)^{-1}(K)$. This completes the proof that

$$(g \circ f)^{-1}(K) = f^{-1}(g^{-1}(K)). \quad \blacksquare$$

We now introduce some special functions: the restriction, inclusion and identity functions.

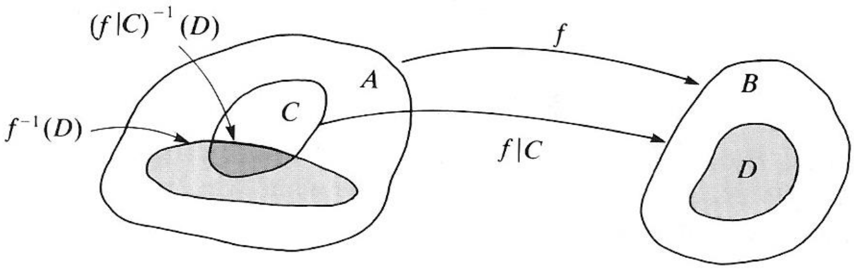
Let C be a subset of a set A. If $f:A \longrightarrow B$ is a function, then the function $x \longmapsto f(x)$, $x \in C$, with domain C and codomain B is called the *restriction* of f to C and denoted by $f|C$. The *inclusion* of C in A is the function $i_C:C \longrightarrow A$ given by $i_C(x) = x$, $x \in C$. If $id_A:A \longrightarrow A$ denotes the *identity function* on the set A, given by $id_A(x) = x$, $x \in A$, then $i_C = id_A|C$. Notice that if $f:A \longrightarrow B$ is a function, then $f|C = f \circ i_C$; in particular, $f \circ id_A = f$. Also if $g:B \longrightarrow A$ is a function then $id_A \circ g = g$.

PROPOSITION 6 If $f:A \longrightarrow B$ is a function, $C \subset A$ and $D \subset B$ then

$$(f|C)^{-1}(D) = C \cap f^{-1}(D).$$

PROOF Since $f|C = f \circ i_C$, where i_C is the inclusion of C in A, it follows from Proposition 5 that

$$(f|C)^{-1}(D) = (f \circ i_C)^{-1}(D) = i_C^{-1}(f^{-1}(D)).$$



But

$$i_C^{-1}(f^{-1}(D)) = \{x : x \in C \text{ and } x \in f^{-1}(D)\} = C \cap f^{-1}(D). \quad \blacksquare$$

A function $f:A \longrightarrow B$ is said to be *injective* if $f(x) = f(y)$ only if $x = y$. Thus f is injective if and only if each element of B is the image of *at most* one element of A. Also $f:A \longrightarrow B$ is said to be *surjective* or an *onto function* if $f(A) = B$. Thus f is an onto function if and only if each element of B is the image of *at least* one element of A.

EXAMPLE 16 (a) The function $f:\mathbf{N} \longrightarrow \mathbf{N}$ given by

$$f(n) = n^2, \quad n \in \mathbf{N},$$

is an injective function. For if $m, n \in \mathbf{N}$ and $m^2 = n^2$ then $m = n$. But f is not an onto function; for if $k \in \mathbf{N}$ then $k \in f(\mathbf{N})$ if and only if k is the square of an integer. Thus, for example, $2 \notin f(\mathbf{N})$.

(b) Let $B = \{y \in \mathbf{R} : y \geq 0\}$. The function $g : \mathbf{R} \longrightarrow B$ given by

$$g(x) = x^2, \quad x \in \mathbf{R},$$

is an onto function, for $y = (\sqrt{y})^2 = g(\sqrt{y})$ if $y \in B$. But g is not an injective function; for if $x \in \mathbf{R}$ and $x \neq 0$ then $x \neq -x$, whilst

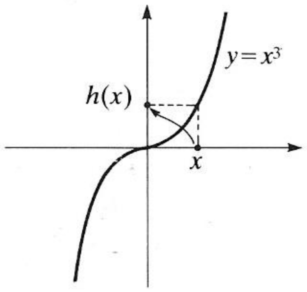
$$g(x) = x^2 = (-x)^2 = g(-x). \quad \square$$

A function $f : A \longrightarrow B$ is said to be *bijective* if it is both injective and an onto function. A bijective function may also be called a *bijection*. Thus $f : A \longrightarrow B$ is a bijection if and only if each element of B is the image of a unique element of A. For this reason, a bijection is sometimes called a *one-one correspondence*.

EXAMPLE 17 The function $h : \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$h(x) = x^3, \quad x \in \mathbf{R},$$

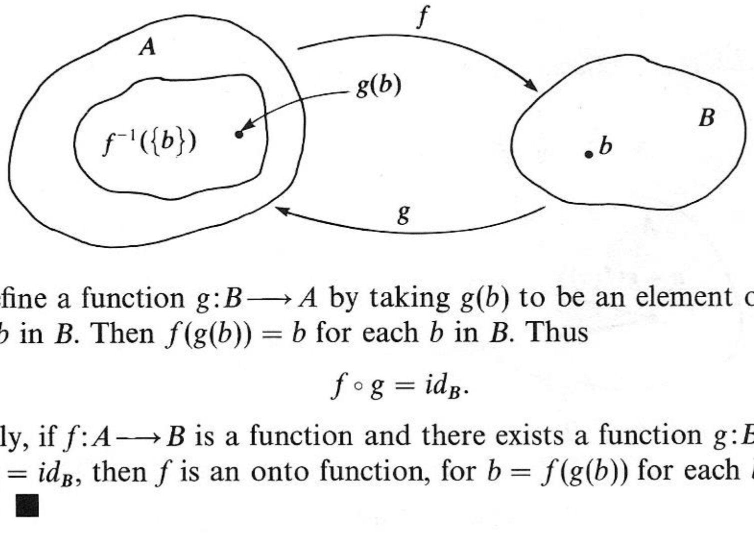
is a bijection. $\square$



The following necessary and sufficient condition for a function to be an onto function will be useful later in this unit.

PROPOSITION 7 A function $f : A \longrightarrow B$ is an onto function if and only if there exists a function $g : B \longrightarrow A$ such that $f \circ g = id_B$.

PROOF Let $f : A \longrightarrow B$ be an onto function. Then for each b in B, there exists at least one element a of A such that $f(a) = b$. Thus for each b in B, the subset $f^{-1}(\{b\})$ of A is non-empty.



Let us define a function $g : B \longrightarrow A$ by taking $g(b)$ to be an element of $f^{-1}(\{b\})$ for each b in B. Then $f(g(b)) = b$ for each b in B. Thus

$$f \circ g = id_B.$$

Conversely, if $f : A \longrightarrow B$ is a function and there exists a function $g : B \longrightarrow A$ such that $f \circ g = id_B$, then f is an onto function, for $b = f(g(b))$ for each b in B and $g(b) \in A$. ■

There is an analogous necessary and sufficient condition for a function to be injective.

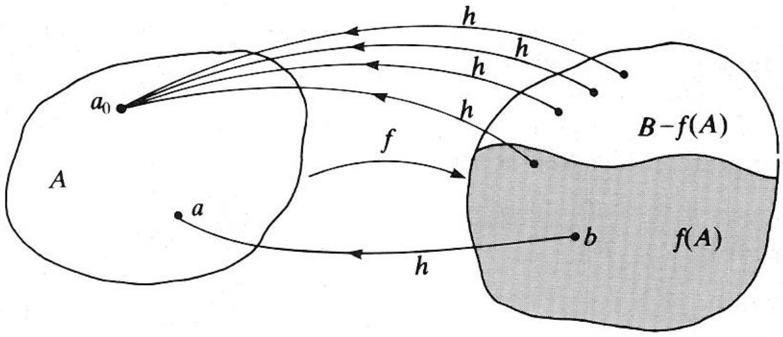
PROPOSITION 8 A function $f : A \longrightarrow B$ is injective if and only if there exists a function $h : B \longrightarrow A$ such that $h \circ f = id_A$.

PROOF Let $f : A \longrightarrow B$ be a function for which there exists $h : B \longrightarrow A$ such that $h \circ f = id_A$. Then $h(f(a)) = a$ for each a in A. Thus if $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ then

$$a_1 = h(f(a_1)) = h(f(a_2)) = a_2.$$

Therefore f is an injective function.

Conversely, let $f:A\longrightarrow B$ be an injective function. If $b\in f(A)$ then there exists a unique element a of A such that $f(a)=b$, whilst if $b\in B-f(A)$ there is no element a of A such that $f(a)=b$. We shall construct a function $h:B\longrightarrow A$ such that $h\circ f=id_A$. Let a_0 be a fixed element of A. For each b in $f(A)$, let $h(b)$ be the unique element of A such that $f(h(b))=b$, and for each b in $B-f(A)$ let $h(b)=a_0$.



We thus obtain a function $h:B\longrightarrow A$. Moreover, for each a in A, $f(a)\in f(A)$ so that $h(f(a))=a$. Hence there exists $h:B\longrightarrow A$ such that $h\circ f=id_A$, as required. ■

Now we can see that the bijections with domain A and codomain B are precisely the functions from A to B which are 'invertible'. By definition, a function is a bijection if and only if it is both injective and onto. Therefore $f:A\longrightarrow B$ is a bijection if and only if there exist $h:B\longrightarrow A$ and $g:B\longrightarrow A$ such that

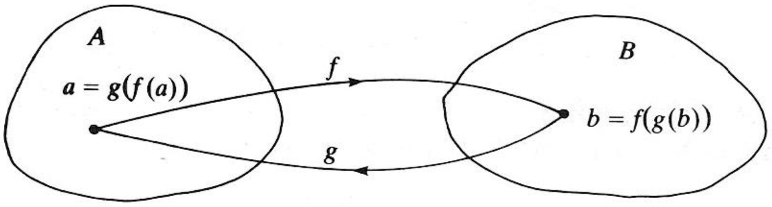
$$h\circ f=id_A \quad\text{and}\quad f\circ g=id_B.$$

But then

$$h=h\circ id_B=h\circ(f\circ g)=(h\circ f)\circ g=id_A\circ g=g.$$

Thus $f:A\longrightarrow B$ is a bijection if and only if there exists a function $g:B\longrightarrow A$ such that

$$g\circ f=id_A \quad\text{and}\quad f\circ g=id_B.$$



The unique function $g:B\longrightarrow A$ such that $g\circ f=id_A$ and $f\circ g=id_B$ is called the *inverse* of the bijection $f:A\longrightarrow B$ and we shall write $g=f^{-1}$. Then

$$f^{-1}(b)=a \text{ if and only if } f(a)=b$$

for all a in A and b in B. Thus for every b in B, $f^{-1}(b)$ is the unique element of A such that $f(f^{-1}(b))=b$.

It will be clear that if $f:A\longrightarrow B$ is a bijection, then $f^{-1}:B\longrightarrow A$ is a bijection and $(f^{-1})^{-1}=f$. Also, if $f:A\longrightarrow B$ and $g:B\longrightarrow C$ are bijections then $g\circ f:A\longrightarrow C$ is a bijection and $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$.

Remark. We should observe that if $f:A\longrightarrow B$ is a bijection and $D\subset B$ then there are apparently two interpretations of $f^{-1}(D)$, namely the inverse image C of D under f and the image E of D under f^{-1}. But if $a\in C$ then $f(a)\in D$ and $a=f^{-1}(f(a))$ so that $a\in E$; and if $a\in E$ then $a=f^{-1}(b)$ where $b\in D$ so that $f(a)=b\in D$ and hence $a\in C$. Thus $C=E$ and the problem of the interpretation of $f^{-1}(D)$ has disappeared.

It should be stressed, however, that if $f:A \longrightarrow B$ is any function, not necessarily a bijection, then for each subset D of B, the subset

$$f^{-1}(D) = \{a : a \in A \text{ and } f(a) \in D\}$$

is well-defined, but it **cannot** be interpreted as the image of D under f^{-1} unless f is a bijection. ☐

A function f with domain a set A and codomain the set $\mathbf{R}$ of real numbers is called a *real-valued function* on A. We often combine real-valued functions. For example, if $f:A \longrightarrow \mathbf{R}$ and $g:A \longrightarrow \mathbf{R}$ are real-valued functions then the real-valued function $f + g$ on A is given by

$$(f+g)(x) = f(x) + g(x), \quad x \in A.$$

We note that $f(x)$ and $g(x)$ are real numbers for each x in A, so that $f(x) + g(x)$ is meaningful. More generally, if s and t are real numbers, we define $sf + tg : A \longrightarrow \mathbf{R}$ by

$$(sf + tg)(x) = sf(x) + tg(x), \quad x \in A.$$

Also, if f and g are real-valued functions on A then $fg : A \longrightarrow \mathbf{R}$ is given by

$$(fg)(x) = f(x)g(x), \quad x \in A.$$

You will recall that if t is a real number then the *modulus* of t, denoted by $|t|$, is defined by

$$|t| = \begin{cases} t, & \text{if } t \geq 0, \\ -t, & \text{if } t < 0. \end{cases}$$

Properties of the modulus function will be studied in *Unit 2*, Section 2.1. If f is a real-valued function on A then the real-valued function $|f|$ is defined by putting

$$|f|(x) = |f(x)|, \quad x \in A.$$

Finally, if f and g are real-valued functions on A we define real-valued functions $\operatorname{Max}\{f, g\}$ and $\operatorname{Min}\{f, g\}$ by putting

$$\operatorname{Max}\{f, g\}(x) = \operatorname{Max}\{f(x), g(x)\}, \quad x \in A,$$

and

$$\operatorname{Min}\{f, g\}(x) = \operatorname{Min}\{f(x), g(x)\}, \quad x \in A.$$

SAQ 12 Let $f:\mathbf{R} \longrightarrow \mathbf{R}$ and $g:\mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$f(x) = 2x, \quad g(x) = x + 1, \quad x \in \mathbf{R}.$$

Show that $g \circ f \neq f \circ g$.

SAQ 13 Let $f:\mathbf{N} \longrightarrow \mathbf{N}$ be given by $f(n) = 3n$, $n \in \mathbf{N}$, and let

$$D = \{x \in \mathbf{N} : x \text{ divides } 60\} \quad \text{and} \quad E = \{x \in \mathbf{N} : x \text{ divides } 70\}.$$

Find $f^{-1}(D)$ and $f^{-1}(E)$.

***SAQ 14** Let $f:A \longrightarrow B$ be a function and let C and D be subsets of A and B respectively. Prove that:

(a) $C \subset f^{-1}(f(C))$;

(b) $C = f^{-1}(f(C))$ if f is injective;

(c) $f(C \cap f^{-1}(D)) = f(C) \cap D$;

(d) $f(f^{-1}(D)) = f(A) \cap D \subset D$;

(e) $f(f^{-1}(D)) = D$ if $D \subset f(A)$.

SAQ 15 Let $f:A \longrightarrow B$ be a function, let $(C_i)_{i \in I}$ be an indexed family of subsets of A and let $(D_j)_{j \in J}$ be an indexed family of subsets of B. Prove that

(a) $f(\bigcup_{i \in I} C_i) = \bigcup_{i \in I} f(C_i)$;

(b) $f(\bigcap_{i \in I} C_i) \subset \bigcap_{i \in I} f(C_i)$;

(c) $f^{-1}(\bigcup_{j \in J} D_j) = \bigcup_{j \in J} f^{-1}(D_j)$;

(d) $f^{-1}(\bigcap_{j \in J} D_j) = \bigcap_{j \in J} f^{-1}(D_j)$.

SAQ 16 Let $f:\mathbf{N} \longrightarrow \mathbf{N}$ be given by $f(n) = 2n$, $n \in \mathbf{N}$. Prove that there exists no function $g:\mathbf{N} \longrightarrow \mathbf{N}$ such that $f \circ g = id_{\mathbf{N}}$ and find two *different* functions $h_1, h_2:\mathbf{N} \longrightarrow \mathbf{N}$ such that $h_1 \circ f = h_2 \circ f = id_{\mathbf{N}}$.

SAQ 17 Let $f:\mathbf{R} \longrightarrow B \subset \mathbf{R}$ be the function defined by putting

$$f(x) = |x| + x, \quad x \in \mathbf{R}.$$

(a) Given that f is an onto function, find B.

(b) Find a subset A of $\mathbf{R}$ such that $f|A:A \longrightarrow B$ is a bijection, where B is as in part (a).

SAQ 18 Let

$$A = \{x \in \mathbf{R}: x \neq -\tfrac{1}{2}\}, \quad B = \{y \in \mathbf{R}: y \neq \tfrac{1}{2}\}$$

and let $f:A \longrightarrow B$ be defined by

$$f(x) = \frac{x-3}{2x+1}, \quad x \in A.$$

Prove that f is a bijection and find its inverse function f^{-1}.

SAQ 19 Let A be the closed interval $[0, 1]$ in $\mathbf{R}$ and let $f, g:A \longrightarrow \mathbf{R}$ be given by

$$f(x) = 2x^2, \quad g(x) = x, \quad x \in A.$$

Find the following real functions on A:(i) $3f + 7g$; (ii) fg; (iii) $|f - g|$; (iv) $\mathrm{Min}\{f, g\}$.

SOLUTION TO SAQ 12 If $x \in \mathbf{R}$ then

$$(g \circ f)(x) = g(f(x)) = f(x) + 1 = 2x + 1,$$

whilst

$$(f \circ g)(x) = f(g(x)) = 2g(x) = 2x + 2. \quad ■$$

SOLUTION TO SAQ 13

$$\begin{aligned} f^{-1}(D) &= \{n \in \mathbf{N}: f(n) \in D\} \\ &= \{n \in \mathbf{N}: 3n \text{ divides } 60\} \\ &= \{1, 2, 4, 5, 10, 20\}. \end{aligned}$$

Since 3 is a prime number which does not divide 70, there is no integer n such that $3n$ divides 70. Hence $f^{-1}(E) = \varnothing$. ■

SOLUTION TO SAQ 14

(a) If $x \in A$ then $x \in f^{-1}(f(C))$ if and only if $f(x) \in f(C)$. If $x \in C$ then $f(x) \in f(C)$, so that $x \in f^{-1}(f(C))$. Thus $C \subset f^{-1}(f(C))$.

(b) If $x \in f^{-1}(f(C))$ then $f(x) \in f(C)$, so there exists c in C such that $f(x) = f(c)$. If f is injective we can deduce that $x = c$ so that $x \in C$. Thus if f is injective then $f^{-1}(f(C)) \subset C$, so that $C = f^{-1}(f(C))$ by (a).

(c) If $x \in C \cap f^{-1}(D)$, then $f(x) \in f(C)$ since $x \in C$ and $f(x) \in D$ since $x \in f^{-1}(D)$. Thus $f(x) \in f(C) \cap D$. Hence $f(C \cap f^{-1}(D)) \subset f(C) \cap D$. If $y \in f(C) \cap D$ then $y = f(x)$ where $x \in C$. But $f(x) = y \in D$, so that $x \in f^{-1}(D)$. Thus $y = f(x)$ where $x \in C \cap f^{-1}(D)$, so that $y \in f(C \cap f^{-1}(D))$. Hence $f(C) \cap D \subset f(C \cap f^{-1}(D))$ and so $f(C) \cap D = f(C \cap f^{-1}(D))$.

(d) This follows from (c) on putting $C = A$.

(e) This is an immediate consequence of (d), since $f(A) \cap D = D$ if $D \subset f(A)$. We note that in particular it follows that $f(f^{-1}(D)) = D$ if f is an onto function. ■

SOLUTION TO SAQ 15

(a)
$$\begin{aligned} y \in f(\bigcup_{i \in I} C_i) &\Longleftrightarrow y = f(x) \text{ where } x \in \bigcup_{i \in I} C_i \\ &\Longleftrightarrow y = f(x) \text{ where } x \in C_i \text{ for some } i \\ &\Longleftrightarrow y \in f(C_i) \text{ for some } i \\ &\Longleftrightarrow y \in \bigcup_{i \in I} f(C_i). \end{aligned}$$

(b) If $x \in \bigcap_{i \in I} C_i$ then $x \in C_i$ for every i so that $f(x) \in f(C_i)$ for every i, and hence $f(x) \in \bigcap_{i \in I} f(C_i)$. Thus $f(\bigcap_{i \in I} C_i) \subset \bigcap_{i \in I} f(C_i)$.

(c)
$$\begin{aligned} x \in f^{-1}(\bigcup_{j \in J} D_j) &\Longleftrightarrow f(x) \in \bigcup_{j \in J} D_j \\ &\Longleftrightarrow f(x) \in D_j \text{ for some } j \\ &\Longleftrightarrow x \in f^{-1}(D_j) \text{ for some } j \\ &\Longleftrightarrow x \in \bigcup_{j \in J} f^{-1}(D_j). \end{aligned}$$

(d)
$$\begin{aligned} x \in f^{-1}(\bigcap_{j \in J} D_j) &\Longleftrightarrow f(x) \in \bigcap_{j \in J} D_j \\ &\Longleftrightarrow f(x) \in D_j \text{ for every } j \\ &\Longleftrightarrow x \in f^{-1}(D_j) \text{ for every } j \\ &\Longleftrightarrow x \in \bigcap_{j \in J} f^{-1}(D_j). \quad ■ \end{aligned}$$

SOLUTION TO SAQ 16

The function f is not an onto function ($f(\mathbf{N})$ is the set of even integers). Hence, by Proposition 7, there is no function $g:\mathbf{N} \longrightarrow \mathbf{N}$ such that $f \circ g = id_{\mathbf{N}}$. [Alternative argument: Suppose there exists $g:\mathbf{N} \longrightarrow \mathbf{N}$ such that $f \circ g = id_{\mathbf{N}}$; then $f(g(1)) = 1$, that is $2g(1) = 1$, which is absurd since $g(1)$ is an integer; hence there is no such function g.] Let $h_1, h_2:\mathbf{N} \longrightarrow \mathbf{N}$ be defined as follows:

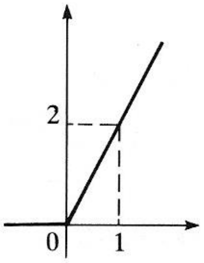
$$h_1(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd,} \end{cases} \qquad h_2(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Since $h_1(2n) = h_2(2n) = n$ for every n in $\mathbf{N}$, it follows that $h_1 \circ f = h_2 \circ f = id_{\mathbf{N}}$. ■

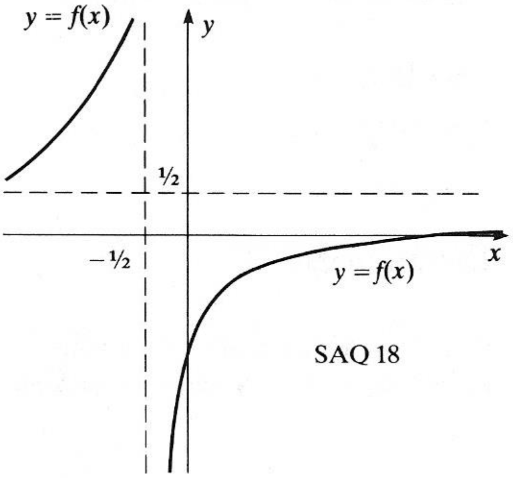
SOLUTION TO SAQ 17

(a) If $x \in \mathbf{R}$, then $f(x) \geq 0$. Also if $t \in \mathbf{R}$ and $t \geq 0$, then $f(\frac{1}{2}t) = t$. Thus $B = \{t \in \mathbf{R}: t \geq 0\}$.

(b) If $t > 0$ then $f(x) = t$ if and only if $x = \frac{1}{2}t > 0$. Thus $f|A$ is a bijection if and only if $A = \mathbf{R}_+ \cup \{x_0\}$ where $x_0 \leq 0$, since $f(x_0) = 0$ if $x_0 \leq 0$. ■



SAQ 17



SAQ 18

SOLUTION TO SAQ 18 If $x \neq -\frac{1}{2}$ then

$$\frac{x-3}{2x+1} = y \Longleftrightarrow x - 3 = (2x+1)y \Longleftrightarrow x = \frac{3+y}{1-2y},$$

provided that $y \neq \frac{1}{2}$.

Thus for each y in B there is precisely one x in A such that $f(x) = y$. Hence f is a bijection. Moreover, for each y in B, $f^{-1}(y)$ is the unique element of A such that $f(f^{-1}(y)) = y$. Thus

$$f^{-1}(y) = \frac{3+y}{1-2y}, \quad y \in B. \quad \blacksquare$$

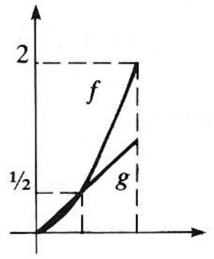
SOLUTION TO SAQ 19

(i) $(3f + 7g)(x) = 6x^2 + 7x, \quad x \in A.$

(ii) $(fg)(x) = 2x^3, \quad x \in A.$

(iii) $$|f - g|(x) = |f(x) - g(x)| = |x(2x-1)| = \begin{cases} x(1-2x), & \text{if } 0 \leq x \leq \frac{1}{2} \\ x(2x-1), & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

(iv) $$\text{Min}\,\{f, g\}(x) = \begin{cases} 2x^2, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ x, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases} \quad \blacksquare$$



SAQ 19

1.4 The Cartesian Product

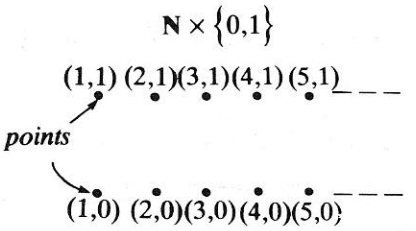
In Section 1.1 we mentioned the representation, by means of Cartesian coordinates, of the plane by the set $\mathbf{R}^2$ of all ordered pairs (x_1, x_2) where $x_1, x_2 \in \mathbf{R}$. If A_1 and A_2 are two arbitrary sets, we can consider ordered pairs (a_1, a_2) where $a_1 \in A_1$ and $a_2 \in A_2$. The ordered pairs (a_1, a_2) and (a_1', a_2') are *equal* if and only if $a_1 = a_1'$ and $a_2 = a_2'$. The *Cartesian product* $A_1 \times A_2$ of sets A_1 and A_2 is defined to be the set consisting of all ordered pairs (a_1, a_2) where $a_1 \in A_1$ and $a_2 \in A_2$. Thus

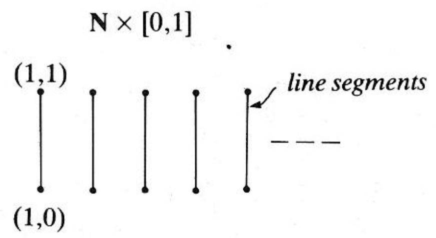
$$A_1 \times A_2 = \{(a_1, a_2) : a_1 \in A_1, a_2 \in A_2\}.$$

The sets A_1, A_2 are called *factors* of $A_1 \times A_2$. Note that $A_1 \times A_2 \neq \varnothing$ if and only if $A_1 \neq \varnothing$ and $A_2 \neq \varnothing$.

EXAMPLE 18 $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$.

EXAMPLE 19 The set $\mathbf{N} \times \{0, 1\}$ consists of all ordered pairs (n, i) where n is a positive integer and $i \in \{0, 1\}$. The set $\mathbf{N} \times [0, 1]$ consists of all ordered pairs (n, t) where $n \in \mathbf{N}$ and t is a real number such that $0 \leq t \leq 1$. These two product sets can be represented as subsets of the plane as follows:

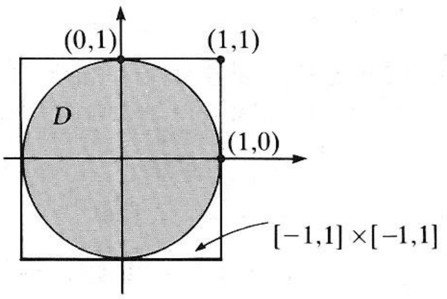




Remark. If A_1 and A_2 are finite sets having m elements and n elements respectively, then $A_1 \times A_2$ is a finite set with mn elements.

EXAMPLE 20 If A_1 and A_2 are non-empty sets and $B_1 \subset A_1, B_2 \subset A_2$, then $B_1 \times B_2 \subset A_1 \times A_2$. Except in the trival case in which one of A_1 and A_2 has just one element, there are subsets of $A_1 \times A_2$ which are not of the form $B_1 \times B_2$ where $B_1 \subset A_1$ and $B_2 \subset A_2$. For example, let

$$D = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 \leq 1\},$$



and suppose that $D = B \times C$ where B and C are subsets of $\mathbf{R}$. Since $(1,0) \in D$ it follows that $1 \in B$, and since $(0,1) \in D$ it follows that $1 \in C$. Therefore $(1,1) \in B \times C = D$. But $1^2 + 1^2 = 2$, so that we have a contradiction. Thus there is no way in which the subset D of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ can be expressed as a product of subsets of $\mathbf{R}$. For example, $[-1,1] \times [-1,1]$ is the square with centre $(0,0)$ and side 2. □

Associated with the Cartesian product $A_1 \times A_2$ of sets A_1 and A_2 are two important functions. The *first projection* $p_1 : A_1 \times A_2 \longrightarrow A_1$ is defined by

$$p_1(x_1, x_2) = x_1, \quad (x_1, x_2) \in A_1 \times A_2,$$

and the *second projection* $p_2 : A_1 \times A_2 \longrightarrow A_2$ is defined by

$$p_2(x_1, x_2) = x_2, \quad (x_1, x_2) \in A_1 \times A_2.$$

We notice that if $x \in A_1 \times A_2$ then

$$x = (p_1(x), p_2(x)). \quad \square$$

The next result will be important in our work in *Unit 4* on Cartesian products of topological spaces.

PROPOSITION 9 Let A_1, A_2 be sets and let

$$p_1 : A_1 \times A_2 \longrightarrow A_1 \quad \text{and} \quad p_2 : A_1 \times A_2 \longrightarrow A_2$$

be the (above) projections. Given a set X and functions

$$f_1 : X \longrightarrow A_1 \quad \text{and} \quad f_2 : X \longrightarrow A_2,$$

there is a unique function $g : X \longrightarrow A_1 \times A_2$ such that $p_1 \circ g = f_1$ and $p_2 \circ g = f_2$.

$$\begin{tikzcd}
A_1 & A_1 \times A_2 \arrow[l, "p_1"'] \arrow[r, "p_2"] & A_2 \\
& X \arrow[ul, "f_1"] \arrow[u, "g"'] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

PROOF If $x \in X$ then $(f_1(x), f_2(x)) \in A_1 \times A_2$. Thus we can define $g : X \longrightarrow A_1 \times A_2$ by

$$g(x) = (f_1(x), f_2(x)), \quad x \in X.$$

Then $p_1(g(x)) = f_1(x)$ for each x in X, so that $p_1 \circ g = f_1$. Similarly, $p_2 \circ g = f_2$.

To establish the uniqueness of g, suppose that $h : X \longrightarrow A_1 \times A_2$ satisfies $p_1 \circ h = f_1$ and $p_2 \circ h = f_2$. Then

$$h(x) = (p_1(h(x)), p_2(h(x))) = (f_1(x), f_2(x)) = g(x)$$

for each x in X, so that $h = g$. ■

If we have functions $f_1 : A_1 \longrightarrow B_1$ and $f_2 : A_2 \longrightarrow B_2$ then we define

$$f_1 \times f_2 : A_1 \times A_2 \longrightarrow B_1 \times B_2$$

by putting

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)), \qquad (x_1, x_2) \in A_1 \times A_2.$$

Notice that if p_1 and p_2 are the first and second projections of $A_1 \times A_2$ and q_1 and q_2 are the first and second projections of $B_1 \times B_2$ then

$$(f_1 \times f_2)(x) = (f_1(p_1(x)), f_2(p_2(x)))$$

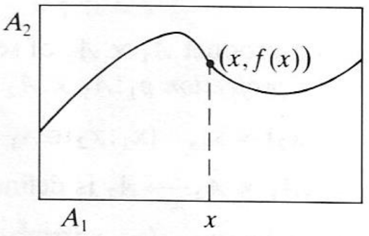
for each x in $A_1 \times A_2$ so that

$$(q_1(f_1 \times f_2))(x) = (f_1 \circ p_1)(x),$$

from which it follows that $q_1 \circ (f_1 \times f_2) = f_1 \circ p_1$. Similarly, $q_2 \circ (f_1 \times f_2) = f_2 \circ p_2$.

We can extend the familiar idea of the graph of a real-valued function defined on an interval in $\mathbf{R}$ to arbitrary functions. If $f : A_1 \longrightarrow A_2$ is a function, then the *graph* of f is the subset G_f of $A_1 \times A_2$ defined by

$$G_f = \{(x_1, x_2) \in A_1 \times A_2 : f(x_1) = x_2\}.$$



The following property of graphs will be of interest in *Unit 5*.

PROPOSITION 10 If $f : A_1 \longrightarrow A_2$ is a function, then there is a bijection between A_1 and the graph G_f of f.

PROOF Let p_1, p_2 be the projections of $A_1 \times A_2$ and let $i : G_f \longrightarrow A_1 \times A_2$ be the inclusion of G_f in $A_1 \times A_2$. Let $\theta : G_f \longrightarrow A_1$ be given by $\theta = p_1 \circ i$ so that $\theta(x_1, x_2) = x_1$ if $(x_1, x_2) \in G_f$. If $x \in A_1$ then $(x, f(x)) \in G_f$ so that we can define a function $\psi : A_1 \longrightarrow G_f$ by putting

$$\psi(x) = (x, f(x)), \quad x \in A_1.$$

Then

$$(\theta \circ \psi)(x) = \theta(\psi(x)) = \theta(x, f(x)) = x$$

for each x in A_1, so that $\theta \circ \psi = id_{A_1}$. Also, if $(x_1, x_2) \in G_f$ then $x_2 = f(x_1)$ and

$$(\psi \circ \theta)(x_1, x_2) = \psi(\theta(x_1, x_2)) = \psi(x_1) = (x_1, f(x_1)) = (x_1, x_2),$$

so that $\psi \circ \theta = id_{G_f}$. Thus ψ is a bijection and $\theta = \psi^{-1}$. ■

Remark. For use in *Unit 5*, we note that if $x \in A_1$ then

$$(p_2 \circ i \circ \psi)(x) = p_2(x, f(x)) = f(x)$$

and hence $p_2 \circ i \circ \psi = f$.

We can extend the idea of the Cartesian product to an arbitrary finite number of factors. If $A_1, A_2, \ldots, A_n$ are sets, then their *Cartesian product*, written $A_1 \times A_2 \times \cdots \times A_n$, is the set consisting of all n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ for $i = 1, \ldots, n$. If $A_i = A$ for $i = 1, \ldots, n$, then we usually denote the product $A_1 \times A_2 \times \cdots \times A_n$ by A^n. We began our discussion of products by citing $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. In the subsequent work we shall often be concerned with $\mathbf{R}^n$. The elements of $\mathbf{R}^n$ are the n-tuples $(x_1, x_2, \ldots, x_n)$ where x_i is a real number for $i = 1, \ldots, n$.

SAQ 20 Let $a = (1, x + y)$ and $b = (x - y, 3)$ be elements of $\mathbf{N} \times \mathbf{N}$. Find x and y if $a = b$.

***SAQ 21** Let A, B and C be sets. Prove that:

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$;

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$;

(c) $A \times (B - C) = (A \times B) - (A \times C)$.

SAQ 22 Let A_1 and A_2 be non-empty sets, let b be a fixed element of A_2 and let $j: A_1 \longrightarrow A_1 \times A_2$ be given by $j(x) = (x, b)$, $x \in A_1$. Show that j is an injective function. Find $j^{-1}(B_1 \times B_2)$ if $B_1 \subset A_1$ and $B_2 \subset A_2$.

SAQ 23 Show that if A_1 and A_2 are non-empty sets then

$$A_1 \times A_2 = \bigcup_{x \in A_1} (\{x\} \times A_2).$$

SAQ 24 Let A_1 and A_2 be non-empty sets and let p_1 and p_2 be the first and second projections respectively of $A_1 \times A_2$. Prove that if $B_1 \times B_2 \subset A_1 \times A_2$ then

$$B_1 \times B_2 = p_1^{-1}(B_1) \cap p_2^{-1}(B_2).$$

Deduce that if X is a set and $f: X \longrightarrow A_1 \times A_2$ is a function then

$$f^{-1}(B_1 \times B_2) = (p_1 \circ f)^{-1}(B_1) \cap (p_2 \circ f)^{-1}(B_2).$$

SOLUTION TO SAQ 20 Since $a = b$ if and only if $x - y = 1$ and $x + y = 3$ we see that $x = 2$, $y = 1$. ■

SOLUTION TO SAQ 21

(a)
$$\begin{aligned}(x, y) \in A \times (B \cup C) &\Longleftrightarrow x \in A,\ y \in B \cup C \\ &\Longleftrightarrow x \in A,\ y \in B \text{ or } y \in C \\ &\Longleftrightarrow (x, y) \in A \times B \text{ or } (x, y) \in A \times C \\ &\Longleftrightarrow (x, y) \in (A \times B) \cup (A \times C).\end{aligned}$$

(b)
$$\begin{aligned}(x, y) \in A \times (B \cap C) &\Longleftrightarrow x \in A,\ y \in B \cap C \\ &\Longleftrightarrow x \in A,\ y \in B \text{ and } y \in C \\ &\Longleftrightarrow (x, y) \in A \times B \text{ and } (x, y) \in A \times C \\ &\Longleftrightarrow (x, y) \in (A \times B) \cap (A \times C).\end{aligned}$$

(c)
$$\begin{aligned}(x, y) \in A \times (B - C) &\Longleftrightarrow x \in A,\ y \in B - C \\ &\Longleftrightarrow x \in A,\ y \in B \text{ and } y \notin C \\ &\Longleftrightarrow (x, y) \in A \times B \text{ and } (x, y) \notin A \times C \\ &\Longleftrightarrow (x, y) \in (A \times B) - (A \times C).\end{aligned}$$ ■

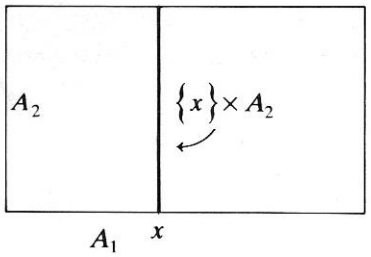
SOLUTION TO SAQ 22 If $x, y \in A_1$ and $j(x) = j(y)$ then $(x, b) = (y, b)$, from which it follows that $x = y$. Thus j is an injective function. If $B_1 \times B_2 \subset A_1 \times A_2$ then

$$\begin{aligned}j^{-1}(B_1 \times B_2) &= \{x \in A_1 : j(x) \in B_1 \times B_2\} \\ &= \{x \in A_1 : (x, b) \in B_1 \times B_2\}.\end{aligned}$$

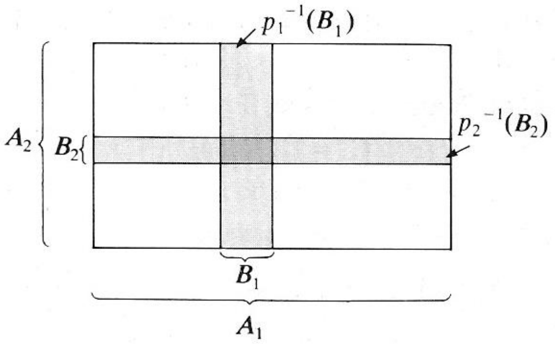
If $b \in B_2$ then $(x, b) \in B_1 \times B_2$ if and only if $x \in B_1$, whilst if $b \notin B_2$ then there is no x in A_1 such that $(x, b) \in B_1 \times B_2$. Thus

$$j^{-1}(B_1 \times B_2) = \begin{cases} B_1, & \text{if } b \in B_2, \\ \varnothing, & \text{if } b \notin B_2. \end{cases}$$ ■

SOLUTION TO SAQ 23 Plainly $\bigcup_{x \in A_1} (\{x\} \times A_2) \subset A_1 \times A_2$. But if $z = (x, y) \in A_1 \times A_2$ then $z \in \{x\} \times A_2$. Thus $\bigcup_{x \in A_1} (\{x\} \times A_2) = A_1 \times A_2$. ■



SAQ 23



SAQ 24

SOLUTION TO SAQ 24

$$\begin{aligned} x\in B_1\times B_2 &\Longleftrightarrow (p_1(x), p_2(x))\in B_1\times B_2\\ &\Longleftrightarrow p_1(x)\in B_1 \text{ and } p_2(x)\in B_2\\ &\Longleftrightarrow x\in p_1^{-1}(B_1) \text{ and } x\in p_2^{-1}(B_2)\\ &\Longleftrightarrow x\in p_1^{-1}(B_1)\cap p_2^{-1}(B_2). \end{aligned}$$

Using the result of SAQ 16 part (d) we have

$$\begin{aligned} f^{-1}(B_1\times B_2) &= f^{-1}(p_1^{-1}(B_1)\cap p_2^{-1}(B_2))\\ &= f^{-1}(p_1^{-1}(B_1))\cap f^{-1}(p_2^{-1}(B_2))\\ &= (p_1\circ f)^{-1}(B_1)\cap (p_2\circ f)^{-1}(B_2), \end{aligned}$$

by Proposition 5. ■

1.5 Countability

A set A is said to be *countably infinite* if there exists a bijection from **N** onto A. Thus a set is countably infinite if and only if its elements can be enumerated as $a_1, a_2, \ldots, a_n, \ldots$ where a_n is the element of A which corresponds to the positive integer n under the bijection between **N** and A. If a set is either finite or countably infinite, it is said to be a *countable set.* By convention, we regard the empty set, $\varnothing$, which has no elements, as a finite set. A set which is not countable is called *uncountable.* Of course, an uncountable set is infinite. Our prime concern here will be with countable sets; however, you should appreciate that uncountable sets do exist, and we give some examples at the end of this section.

EXAMPLE 21 The subset

$$A = \{2, 3, 4, \ldots\}$$

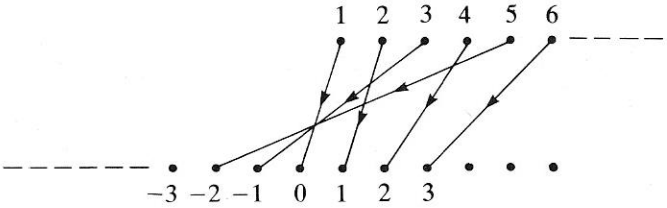
of **N** is countably infinite; for $n \longmapsto n+1$, $n\in\mathbf{N}$, is a bijection from **N** onto A. This illustrates a surprising property of infinite sets (which we mention without proof): a set is infinite if and only if it can be put in one-one correspondence with a proper subset of itself. The set E of even positive integers is countably infinite, for $n \longmapsto 2n$, $n\in\mathbf{N}$, is a bijection from **N** onto E and the set O of odd positive integers is countably infinite, for $n \longmapsto 2n-1$, $n\in\mathbf{N}$, is a bijection from **N** onto O.

Remark. Every infinite subset B of **N** is countably infinite. For let n_1 be the least member of B, n_2 the least member of $B-\{n_1\}$ (so that $n_2 > n_1$), n_3 the least member of $B-\{n_1, n_2\}$ (so that $n_3 > n_2$), and so on. Since B is infinite this process can be continued indefinitely. In this way it is fairly clear that we can write B as

$$B = \{n_1, n_2, \ldots, n_i, n_{i+1}, \ldots\},$$

where $n_{i+1} > n_i$ for each i. Then $i \longmapsto n_i$, $i\in\mathbf{N}$, is a bijection from **N** onto B.

XAMPLE 22 The set $\mathbf{Z}$ of all integers is countably infinite. For example, we can set up a bijection between $\mathbf{N}$ and $\mathbf{Z}$ by mapping the even positive integers bijectively onto the positive integers and mapping the odd positive integers bijectively onto the non-positive integers:



More formally, if for each positive integer n we put $f(2n-1) = -n+1$ and $f(2n) = n$ then we obtain a bijection $f:\mathbf{N} \longrightarrow \mathbf{Z}$.

We note that there will be many (in fact, infinitely many) different bijections of $\mathbf{N}$ onto a given countably infinite set. In this case, we obtain a second bijection g of $\mathbf{N}$ onto $\mathbf{Z}$ by putting $g(2n-1) = n$ and $g(2n) = -n+1$ for each positive integer n. □

Since the composition of bijections is a bijection, it is easy to see that if $f:A \longrightarrow B$ is a bijection and either A or B is countably infinite then so is the other.

OPOSITION **11** (a) If A is an infinite set and $f:A \longrightarrow \mathbf{N}$ is an injective function then A is a countably infinite set.

(b) Every subset of a countably infinite set is countable.

PROOF (a) Since $f(A)$ is an infinite subset of $\mathbf{N}$, it is countably infinite. But the function $a \longmapsto f(a)$, $a \in A$, with domain A and codomain $f(A)$, is a bijection so that A is a countably infinite set.

(b) Let A be a countably infinite set and let B be an infinite subset of A. We must show that B is countably infinite. Since A is countably infinite, there exists a bijection $f:A \longrightarrow \mathbf{N}$. Since $f|B$ is injective, it follows from (a) that B is countably infinite. ■

AMPLE 23 The Cartesian product $\mathbf{N} \times \mathbf{N}$ is a countably infinite set. We shall prove this by constructing an injection $f:\mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$. There are many possibilities; one is as follows.

Let $f:\mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$ be defined by $f(m, n) = 2^m 3^n$, $(m, n) \in \mathbf{N} \times \mathbf{N}$. From the uniqueness of prime factorization, it follows that if m, n, p and q are positive integers and $2^m 3^n = 2^p 3^q$ then $m = p$ and $n = q$; hence f is indeed an injective function. It follows from Proposition 11(a) that $\mathbf{N} \times \mathbf{N}$ is a countably infinite set.

Furthermore, if A_1 and A_2 are countably infinite sets then $A_1 \times A_2$ is a countably infinite set, for if $f_1:\mathbf{N} \longrightarrow A_1$ and $f_2:\mathbf{N} \longrightarrow A_2$ are bijections then

$$f_1 \times f_2 : \mathbf{N} \times \mathbf{N} \longrightarrow A_1 \times A_2$$

is a bijection.

It follows, by using mathematical induction (for example), that any finite product of countably infinite sets is a countably infinite set. □

There is another useful criterion for a set to be countably infinite.

OPOSITION **12** If A is a countably infinite set, B is an infinite set and $f:A \longrightarrow B$ is an onto function, then B is a countably infinite set.

PROOF Since $f: A \longrightarrow B$ is an onto function, it follows from Proposition 7 that there exists a function $g: B \longrightarrow A$ such that $f \circ g = id_B$. But, by Proposition 8, the existence of f such that $f \circ g = id_B$ implies that g is an injective function. Thus we have an injective function $g: B \longrightarrow A$, so that B is in one-one correspondence with an infinite subset of A and hence B is a countably infinite set. ■

EXAMPLE 24 Now we can obtain the perhaps unexpected result that the set $\mathbf{Q}$ of rational numbers is a countably infinite set. If we define

$$f: \mathbf{Z} \times (\mathbf{Z} - \{0\}) \longrightarrow \mathbf{Q}$$

by

$$f(m, n) = \frac{m}{n} \quad \text{if } m, n \in \mathbf{Z},\ n \neq 0,$$

then f is an onto function. Since $\mathbf{Z} \times (\mathbf{Z} - \{0\})$ is a countably infinite set (Example 23) it follows from Proposition 12 that $\mathbf{Q}$ is a countably infinite set. □

As a second application of Proposition 12 we prove:

PROPOSITION **13** The union of a countable family of countably infinite sets is a countably infinite set.

PROOF Let $A = \bigcup_{n=1}^{\infty} A_n$, where each set A_n is countably infinite. For each n in $\mathbf{N}$, there exists a bijection $f_n: \mathbf{N} \longrightarrow A_n$. Now consider $g: \mathbf{N} \times \mathbf{N} \longrightarrow A$ given by

$$g(m, n) = f_n(m), \quad (m, n) \in \mathbf{N} \times \mathbf{N}.$$

If $x \in A$ then $x \in A_n$ for some n in $\mathbf{N}$, and so there exists some m in $\mathbf{N}$ such that $f_n(m) = x$. Thus $g(m, n) = x$. Therefore g is an onto function. Since $\mathbf{N} \times \mathbf{N}$ is a countably infinite set, it follows from Proposition 12 that A is a countably infinite set. ■

EXAMPLE 25 Finally, we give an example of an uncountable set. Consider the set $2^{\mathbf{N}}$ of all subsets of the set $\mathbf{N}$ of positive integers and suppose that it is a countably infinite set. Then there is a one-one correspondence $n \longmapsto A_n$ between $\mathbf{N}$ and $2^{\mathbf{N}}$.

A given positive integer n may or may not be an element of the subset A_n with which it is associated. Now consider the set B defined by the property that, for every positive integer n, $n \in B$ if and only if $n \notin A_n$. Thus if, say, $1 \in A_1$ then $1 \notin B$, and if $7 \notin A_7$ then $7 \in B$. Since $B \in 2^{\mathbf{N}}$ we have $B = A_m$ for some positive integer m. But this is absurd since either

(i) $m \in A_m$ when (by definition) $m \notin B$, which contradicts the assertion $B = A_m$; or

(ii) $m \notin A_m$ when (by definition) $m \in B$, which again contradicts the assertion $B = A_m$.

This contradiction shows that $2^{\mathbf{N}}$ is not countably infinite and hence is an uncountable set.

Remark. The preceding result can be used to show that the set $\mathbf{R}$ of real numbers is an uncountable set.

SAQ 25 Show that the subset of $2^{\mathbf{R}}$ consisting of all intervals $[n, n+1]$ where $n \in \mathbf{Z}$ is countably infinite.

SAQ 26 Let E be the subset of the plane consisting of all points of the plane both coordinates of which are rational. Prove that E is a countably infinite set. Deduce that the set of circles in the plane with centre at a point of E and rational radius is countably infinite.

SOLUTION TO SAQ 25 Let

$$B = \{X \in 2^{\mathbf{R}} : X = [n, n+1] \text{ for some } n \in \mathbf{Z}\}.$$

Then we have a bijection from $\mathbf{Z}$ onto B given by

$$n \longmapsto [n, n+1], \quad n \in \mathbf{Z}.$$

Since $\mathbf{Z}$ is countably infinite, it follows that B is countably infinite. ■

SOLUTION TO SAQ 26 Since $E = \mathbf{Q} \times \mathbf{Q}$ it is clear that E is countably infinite. Let T denote the set of positive rational numbers. For each z in E and each t in T let $S(z; t)$ denote the circle centre z and radius t. If F denotes the set of circles with centre at a point of E and rational radius, then

$$(z, t) \longmapsto S(z, t), \quad (z, t) \in E \times T,$$

is a bijection from $E \times T$ onto F. Since $E \times T$ is a countably infinite set, F is a countably infinite set. ■

1.6 Epilogue (non-assessed)

In this Epilogue we are going to tell a short story which illustrates some of the unexpected properties of infinite sets. We hope you will not think it too frivolous. It is inspired by an extract from a delightful book entitled *Stories about Sets* by N. Ya. Vilenkin (Academic Press, 1968), which can be highly recommended for entertaining and instructive reading.

The Extraordinary Hotel

The Extraordinary Hotel has a countably infinite set of rooms which, naturally, are numbered by the positive integers. The manager of the hotel is the hero of our tale and its villain is the owner. We are going to tell how the manager dealt with a great crisis in his career, but first let us look at some of his routine problems. The owner demands that a guest is never turned away even if the hotel is full. The manager deals with this situation by moving the guest in room 1 to room 2, the guest in room 2 to room 3 and so on, which leaves room 1 vacant for the newcomer. One of the hotel's specialities is the staging of congresses for which there are countably infinite sets of delegates. The owner will accept such a congress even when the hotel is full, leaving the manager to cope with the accommodation. This he usually achieves in the following way. The guest in room 1 is moved to room 2, the guest in room 2 is moved to room 4 and in general the guest in room n is moved to room $2n$. Then the first congress delegate to arrive is put in room 1, the second in room 3 and in general the nth delegate is put in room $2n - 1$.

Now we come to the main part of our story. The hotel used to be known as the First Extraordinary Hotel, being one of a countably infinite chain of similar hotels, naturally known as Second Extraordinary, Third Extraordinary and so on. Some years ago, in a period of financial stringency, the owner decided to close all the hotels except the 'First'. All the hotels were fully booked for the date fixed for the closure but the owner offered alternative accommodation at the 'First' for everyone. The manager had to produce a contingency plan. Knowing a little about prime numbers, his first suggestion was to offer room $2^m 3^n$ at the 'First' to the person who had been offered room n at the mth hotel. The owner rejected this scheme because too many rooms would be left vacant. As you might guess from his successful day-to-day running of the hotel, the manager had trained as a mathematician in his youth. Fortunately, he recalled that $\mathbf{N} \times \mathbf{N}$ is a countably infinite set and was able to construct a one-one correspondence $f : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{N}$. Thus he accommodated a countably infinite set of countably infinite sets of guests by allocating room $f(m, n)$ to the person who had reserved room n at the mth hotel. I think we can be confident that our resourceful hero will continue to prosper and in time will rise to the top in the management of countably infinite hotels!

1.7 Objectives of Unit 1

After studying the unit, you should

(1) understand the basic terminology of set theory;

(2) be able to prove that two sets are *equal*, for example by showing that $A \subset B$ and $B \subset A$, or by 'if and only if' arguments;

(3) understand what set unions and intersections are, and be able to determine them explicitly;

(4) be able to illustrate relations between sets using Venn diagrams;

(5) know and be able to use De Morgan's Laws;

(6) know what an inverse image and an inverse function are, how to determine each, and fully understand the relationship between them;

(7) understand the terms *restriction, inclusion, identity function* id_A, *injective, surjective, bijective*, and know the connections between them;

(8) know what a Cartesian product of sets is, and that a subset of $\mathbf{R}^n$ may or may not represent a Cartesian product;

(9) know the basic properties of countable sets, and be able to show that a given set is or is not countable.

1.8 Revision Questions

SAQ 27 Let $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ be indexed families of sets. Prove that:

(a) $\left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} B_j\right) = \bigcup_{(i,j) \in I \times J} (A_i \cap B_j)$;

(b) $\left(\bigcap_{i \in I} A_i\right) \cup \left(\bigcap_{j \in J} B_j\right) = \bigcap_{(i,j) \in I \times J} (A_i \cup B_j)$;

(c) $\left(\bigcup_{i \in I} A_i\right) \times \left(\bigcup_{j \in J} B_j\right) = \bigcup_{(i,j) \in I \times J} (A_i \times B_j)$;

(d) $\left(\bigcap_{i \in I} A_i\right) \times \left(\bigcap_{j \in J} B_j\right) = \bigcap_{(i,j) \in I \times J} (A_i \times B_j)$.

***SAQ 28** Let $(A_n)_{n \in \mathbf{N}}$ and $(B_n)_{n \in \mathbf{N}}$ be indexed families of sets and let $A = \bigcap_{n=1}^{\infty} A_n$ and $B = \bigcap_{n=1}^{\infty} B_n$. Prove that

$$A \cup B \subset \bigcap_{n=1}^{\infty} (A_n \cup B_n).$$

Show that, in general, the inclusion sign cannot be replaced by equality. Prove that if, in addition, $A_{n+1} \subset A_n$ and $B_{n+1} \subset B_n$ for every n, then

$$A \cup B = \bigcap_{n=1}^{\infty} (A_n \cup B_n).$$

***SAQ 29** Let $f:A \longrightarrow B$ be a function, let C_1, C_2 be subsets of A and let D_1, D_2 be subsets of B. Prove that

$$f(C_1) - f(C_2) \subset f(C_1 - C_2), \quad f^{-1}(D_1 - D_2) = f^{-1}(D_1) - f^{-1}(D_2).$$

Show that in the first formula, the inclusion sign cannot, in general, be replaced by equality.

***SAQ 30** If B is a subset of a set A, the *characteristic function of B* is the function $\chi_B: A \longrightarrow \{0,1\}$ given by

$$\chi_B(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \in A - B. \end{cases}$$

Let F be the set of functions with domain A and codomain $\{0,1\}$. Show that the function $\theta: 2^A \longrightarrow F$ defined by

$$\theta(B) = \chi_B, \quad B \in 2^A$$

is a bijection between 2^A and F.

***SAQ 31** Show that the subset of $2^{\mathbf{N}}$ which consists of all non-empty finite subsets of $\mathbf{N}$ is a countably infinite set.

SOLUTION TO SAQ 27

(a)
$$\begin{aligned} x \in (\bigcup A_i) \cap (\bigcup B_j) &\Longleftrightarrow x \in A_i \text{ for some } i \text{ and } x \in B_j \text{ for some } j \\ &\Longleftrightarrow x \in A_i \cap B_j \text{ for some } (i,j) \in I \times J \\ &\Longleftrightarrow x \in \bigcup (A_i \cap B_j). \end{aligned}$$

(b) If $x \in (\bigcap A_i) \cup (\bigcap B_j)$ then either $x \in A_i$ for every i or $x \in B_j$ for every j, so that $x \in A_i \cup B_j$ for every i in I and j in J, and hence $x \in \bigcap (A_i \cup B_j)$. If $x \notin (\bigcap A_i) \cup (\bigcap B_j)$ then $x \notin \bigcap A_i$ and $x \notin \bigcap B_j$ so there exists i_0 in I such that $x \notin A_{i_0}$ and there exists j_0 in J such that $x \notin B_{j_0}$. Thus $x \notin A_{i_0} \cup B_{j_0}$ so that $x \notin \bigcap (A_i \cup B_j)$. Thus $(\bigcap A_i) \cup (\bigcap B_j) = \bigcap (A_i \cup B_j)$.

(c)
$$\begin{aligned} (x,y) \in (\bigcup A_i) \times (\bigcup B_j) &\Longleftrightarrow x \in \bigcup A_i, \ y \in \bigcup B_j \\ &\Longleftrightarrow x \in A_i \text{ for some } i, \ y \in B_j \text{ for some } j \\ &\Longleftrightarrow (x,y) \in A_i \times B_j \text{ for some } i, j \\ &\Longleftrightarrow (x,y) \in \bigcup (A_i \times B_j). \end{aligned}$$

(d)
$$\begin{aligned} (x,y) \in (\bigcap A_i) \times (\bigcap B_j) &\Longleftrightarrow x \in \bigcap A_i, \ y \in \bigcap B_j \\ &\Longleftrightarrow x \in A_i \text{ for every } i, \ y \in B_j \text{ for every } j \\ &\Longleftrightarrow (x,y) \in A_i \times B_j \text{ for every } i,j \\ &\Longleftrightarrow (x,y) \in \bigcap (A_i \times B_j). \quad \blacksquare \end{aligned}$$

SOLUTION TO SAQ 28

If $x \in A \cup B$ then $x \in A$ or $x \in B$; so that $x \in A_n$ for every n or $x \in B_n$ for every n, from which it follows that $x \in A_n \cup B_n$ for every n. Therefore

$$A \cup B \subset \bigcap_{n=1}^{\infty} (A_n \cup B_n).$$

Let $A_1 = \{1\}$ and $A_n = \{2\}$ for all $n \geq 2$ whilst $B_1 = \{2\}$ and $B_n = \{1\}$ for all $n \geq 2$. Then $A = B = \varnothing$ so that $A \cup B = \varnothing$. But $A_n \cup B_n = \{1,2\}$ for every n, and hence $\bigcap_{n=1}^{\infty} (A_n \cup B_n) = \{1,2\}$.

Now suppose that $A_{n+1} \subset A_n$ and $B_{n+1} \subset B_n$ for every n. If $x \notin A \cup B$ then $x \notin A$ and $x \notin B$, so there exist positive integers r, s such that $x \notin A_r$ and $x \notin B_s$. Let $m = \operatorname{Max}\{r,s\}$. Then $A_m \subset A_r$ and $B_m \subset B_s$, so that $x \notin A_m \cup B_m$. Thus $x \notin \bigcap_{n=1}^{\infty} (A_n \cup B_n)$. It now follows that

$$A \cup B = \bigcap_{n=1}^{\infty} (A_n \cup B_n). \quad \blacksquare$$

SOLUTION TO SAQ 29 If $y \in f(C_1) - f(C_2)$ then $y \in f(C_1)$ and $y \notin f(C_2)$. Thus $y = f(x)$ where $x \in C_1$, and since $y \notin f(C_2)$ it follows that $x \notin C_2$. Thus $y = f(x)$ where $x \in C_1 - C_2$ and so $y \in f(C_1 - C_2)$. Hence $f(C_1) - f(C_2) \subset f(C_1 - C_2)$. If $A = \{a_1, a_2\}$, $B = \{b\}$, f is the unique function with domain A and codomain B ($a_i \longmapsto b$, $i = 1, 2$) and $C_1 = \{a_1\}$, $C_2 = \{a_2\}$, then $f(C_1) = B$ and $f(C_2) = B$ so that $f(C_1) - f(C_2) = \varnothing$; whilst $C_1 - C_2 = C_1$ so that $f(C_1 - C_2) = B$. Thus the inclusion sign in the formula just obtained cannot in general be replaced by equality.

To prove the second formula, we observe that

$$\begin{aligned} x \in f^{-1}(D_1 - D_2) &\Longleftrightarrow f(x) \in D_1 - D_2 \\ &\Longleftrightarrow f(x) \in D_1 \text{ and } f(x) \notin D_2 \\ &\Longleftrightarrow x \in f^{-1}(D_1) \text{ and } x \notin f^{-1}(D_2) \\ &\Longleftrightarrow x \in f^{-1}(D_1) - f^{-1}(D_2). \quad \blacksquare \end{aligned}$$

SOLUTION TO SAQ 30 If $B_1, B_2 \in 2^A$ and $\theta(B_1) = \theta(B_2)$, then $\chi_{B_1}(x) = \chi_{B_2}(x)$ for every x in A. If $x \in B_1$ then $\chi_{B_1}(x) = 1$ so that $\chi_{B_2}(x) = 1$ and hence $x \in B_2$. Similarly, if $x \in B_2$ then $x \in B_1$. Thus $B_1 = B_2$. Therefore θ is an injective function. If $f \in F$, let $B = \{x \in A : f(x) = 1\}$. If $x \in B$ then $\chi_B(x) = 1 = f(x)$ and if $x \in A - B$ then $\chi_B(x) = 0 = f(x)$. Thus $\theta(B) = \chi_B = f$. Hence θ is an onto function. Thus we have established that θ is a bijection. ■

SOLUTION TO SAQ 31 Let the set of non-empty finite subsets of $\mathbf{N}$ be denoted by $\mathscr{A}$. Then $\mathscr{A} = \bigcup_{m \in \mathbf{N}} \mathscr{A}_m$, where $\mathscr{A}_m$ is the set of subsets of $\mathbf{N}$ which have precisely m elements. Since a countable union of countably infinite sets is countably infinite, it will be enough to show that $\mathscr{A}_m$ is countably infinite for each positive integer m. First we observe that $\mathbf{N}^m$ is a countably infinite set. Next we construct an injective function $\theta : \mathscr{A}_m \longrightarrow \mathbf{N}^m$ in the following way. If $A \in \mathscr{A}_m$ and $A = \{i_1, i_2, \ldots, i_m\}$ where $i_1 < i_2 < \cdots < i_m$, we put

$$\theta(A) = (i_1, i_2, \ldots, i_m)$$

It is easy to see that if $A, B \in \mathscr{A}_m$ and $A \neq B$ then $\theta(A) \neq \theta(B)$. Thus there is a one-one correspondence between $\mathscr{A}_m$ and a subset of $\mathbf{N}^m$. Hence $\mathscr{A}_m$ is a countably infinite set. ■

UNIT 2
SOME REAL ANALYSIS

2.0 Introduction

Strong motivation for the concepts you will meet in your study of metric and topological spaces in *Units 3–8* comes from (real) analysis. In this unit, therefore, we review the relevant parts of analysis, following **S**.

The approach via *nests* to some concepts in analysis adopted in M203 differs in some respects from that outlined in **S**. In order to make a smooth transition between M203 and M335, we include in this unit many notes on the reading passages in **S** intended to illustrate the relationships between the two developments. So if you were an M203 student you will certainly wish to read these notes, indicated by the symbol *M203* in the left-hand margin. They are, naturally, non-assessed.

If your previous study of analysis was in M231 or any course other than M203, you should ignore these notes intended for M203 students. They will generally be quite unintelligible to you!—and are non-assessed.

M203 In this unit we shall be discussing sequences, limits and continuity using the so-called 'ε-N and ε-δ definitions' of these which you met in an optional section at the end of *Unit 30* of M203. It is *essential* that you master these new (but equivalent) definitions and do *not* attempt to get by (even initially) by using the nests approach of M203. In particular, do *not* solve any SAQs using nests, but instead use the methods of ε-N and ε-δ that you meet in this unit.

2.1 The Real Numbers

The main properties of the set **R** of real numbers are reviewed in this section. For example, we meet the completeness of **R**, the Archimedean property, the rational density property and the triangle inequality.

READ S all of Section 1.1.

NOTES

(i) *Page 1, lines -13 to -9*

M203 The axioms for the real numbers to which **S** refers were listed in M203, *Unit 8*. Axioms A1–A16 there are the ordered field axioms. The completeness axiom is discussed in more detail in Note (iv), below.

(ii) *Page 2, Example 1.1.1 (c)*

Here **S** refers to the half-open interval $(0, 1]$. His notation for intervals, given on **S** page xiii, denotes the open interval with end-points a and b ($a < b$) by the symbol (a, b), whereas M203 and many other books use the symbol $]a, b[$, with corresponding modifications for half-open intervals. Admittedly, there is a slight risk of confusion between the open interval (a, b) in **R** and the point (a, b) in $\mathbf{R}^2$, but in context this never causes trouble. We shall follow the notation of **S**.

(iii) *Page 2, Definition 1.1.2*

The condition that a real number u is the least upper bound of a non-empty subset S of **R** can also be put in the equivalent form:

(a) $s \leq u$ for every s in S; and

(b) if v is any real number smaller than u, then there is at least one s in S with $s > v$.

In other words, (a) ensures that u is *an* upper bound for S and (b) ensures that no *smaller* number can be an upper bound for S. Notice too that every real number larger than u is *an* upper bound for S.

(iv) *Page 2, line* -7

It is important to remember that the least upper bound u of a set S may or may not be in S.

For example, the least upper bound of *both* the half-open interval $(a, b]$ and the open interval (a, b) is b.

(v) *Page 3, Axiom 1.1.4*

M203

The completeness axiom, Axiom 1.1.4, takes a different form from Axiom A17 of M203 *Unit 8*, page 36. The approach of both **S** and M203 *Unit* 8, is that the real number system is 'there' and that the axioms are simply a convenient list of properties from which we can deduce all other facts that are required about numbers. In the M203 development it is shown in *Unit 8*, Section 8.3 (Remarks, page 44) that **R** has the property described in **S** Axiom 1.1.4. We now check the converse result; namely that, *given* the axioms of **S**, we can *prove* that **R** has the property described in M203, Axiom A17.

To this end, let (I_n) be a sequence of nested closed intervals such that $|I_n| \leq \frac{|I_0|}{2^n}$ for every $n \in \mathbf{N}$. For each n, I_n is a closed interval; so we denote it by $I_n = [a_n, b_n]$. Notice that, for each n and m, $a_n \leq b_m$. This follows because, if $n \leq m$, then $I_m \subset I_n$, giving $a_n \leq a_m \leq b_m$; whilst, if $n > m$, then $I_n \subset I_m$ giving $a_n \leq b_n \leq b_m$. In either case, $a_n \leq b_m$.

Now let S be the set of left end-points a_n; that is, $S = \{a_n : n \in \mathbf{N}\}$. S is non-empty and is bounded above: in fact, each of the numbers b_m is an upper bound for S. It follows from Axiom 1.1.4 that the set S has a least upper bound, c say.

Next, let T be the set of right end-points b_m; that is, $T = \{b_m : m \in \mathbf{N}\}$. Then since each b_m is *an* upper bound for S, $c \leq b_m$ for each m (since c is the *least* upper bound), so that c is a lower bound of the non-empty set T. It follows from **S** Proposition 1.1.5 that T has a greatest lower bound, d say, and that $c \leq d$.

We now wish to show that $\bigcap_{n=0}^{\infty} I_n = [c, d]$. For each integer $n \geq 0$, we have

$$a_n \leq c \leq d \leq b_n,$$

so that

$$[c, d] \subset \bigcap_{n=0}^{\infty} I_n.$$

For the reverse inclusion, note that if $x \in \bigcap_{n=0}^{\infty} I_n$, then $a_n \leq x \leq b_n$ for each $n \geq 0$. It follows that x is an upper bound for S so that $x \geq c$, and x is a lower bound for T so that $x \leq d$; in other words, $x \in [c, d]$ so that

$$\bigcap_{n=0}^{\infty} I_n \subset [c, d].$$

This proves the desired equality.

To complete the proof of the assertion of Axiom A17, we now need only show that $c = d$, so that

$$\bigcap_{n=0}^{\infty} I_n = \{c\}.$$

We know that $c \leq d$, so let us suppose that in fact $c < d$, and show that this assumption leads to a contradiction. If $c < d$, then $d - c > 0$; hence by **S** Proposition 1.1.6 there exists a positive integer n such that $n > |I_0|/(d - c)$. Since $2^n > n$ if $n \in \mathbf{N}$, it follows that

$$2^n > |I_0|/(d - c);$$

consequently, $d - c > |I_0|/2^n$

$$\geq |I_n| \quad \text{(by hypothesis of A17).}$$

But since $[c, d] \subset I_n$ we must have $d - c \leq |I_n|$; as this inequality is inconsistent with the previous inequality, we have obtained a contradiction. Therefore $c = d$, and the sequence (I_n) of nested closed intervals indeed does converge to a single point, c. This is the assertion of Axiom A17, and so we are done.

(vi) *Page 3, Proposition 1.1.5*
A real number l is called a *lower bound* of a non-empty subset S of **R** if $l \leq x$ for each x in S. If a lower bound of S exists, we say that S is *bounded below*. Given a non-empty subset S of **R** which is bounded below, we call m a *greatest lower bound* of S if:

(a) m is a lower bound of S; and

(b) $y \leq m$ for any lower bound y of S.

In other words, m is a lower bound of S and there is no greater lower bound of S.

(vii) *Page 3, Proof of Proposition 1.1.5—in detail*
We are given a non-empty subset S of **R** which is bounded below and we must show that S has a greatest lower bound. So let $T = \{x \in \mathbf{R}\colon -x \in S\}$. If m is a lower bound of S then $m \leq s$ for all s in S; so that if $x \in T$ then $m \leq -x$ and hence $x \leq -m$. Thus T is bounded above and $-m$ is an upper bound of T. It follows from **S** Axiom 1.1.4 that T has a least upper bound, u say; so that $x \leq u \leq -m$ for each x in T. Hence $-u \leq -x$, so that $-u$ is a lower bound for S.

But if m is any lower bound for S, then $-m$ is an upper bound for T so that (by definition of least upper bound) $u \leq -m$, and hence $m \leq -u$. Hence $-u$ is the greatest lower bound of S.

(viii) *Page 4, Proposition 1.1.6*
M203 In the development of **R** in M203, this Archimedean property of the real numbers (**S** Proposition 1.1.6) and the fact that between any two distinct real numbers there is a rational number (**S** Proposition 1.1.7), are proved on page 39 of *Unit 8*.

(ix) *Page 4, Proposition 1.1.6*
The following consequence of **S**'s Archimedean property is often used: *Let ε be any positive real number. Then there exists a positive integer N such that $\frac{1}{N} < \varepsilon$.*

For, since $\varepsilon > 0$ then also $\frac{1}{\varepsilon} > 0$. But since (by **S** Proposition 1.1.6) $\frac{1}{\varepsilon}$ cannot be an upper bound for **N** there must exist a positive integer N such that $N > \frac{1}{\varepsilon}$, from which it follows that $\frac{1}{N} < \varepsilon$.

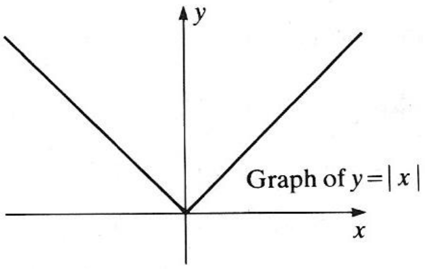
(x) *Page 4, Corollary 1.1.7*
This is often called the *rational density property* of **R**. We shall meet one of its consequences in Note (iv) on the next reading passage.

(xi) *Page 5, lines 1 and 9*
Do not worry about why **S** decided to choose the numbers $(u^2 - 2)/2u$ and $(2 - u^2)/4u$. Simply notice that they give the desired result, and do not spend too much time on the details on page 5 of the Proof of Example 1.1.8.

(xii) *Page 5, Proposition 1.1.9*

Recall that for all $x \in \mathbf{R}$ the *modulus*, $|x|$, of x is the non-negative real number given by

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$



Graph of $y = |x|$

Rather loosely, we can express the formula for $|x|$ and the fact that $|-x| = |x|$ for all $x \in \mathbf{R}$, as that the modulus of x, $|x|$, is *the magnitude of x without its sign.*

Notice that, for all $x \in \mathbf{R}$ and all $t \geq 0$,

$$|x| \leq t \Longleftrightarrow -t \leq x \leq t \qquad (*)$$

For if $|x| \leq t$ then $0 \leq x \leq t$ if $x \geq 0$, while if $x < 0$ then $-x \leq t$ so that $-t \leq x < 0$; so, in both cases, $-t \leq x \leq t$. Conversely, if $-t \leq x \leq t$ then $|x| = x \leq t$ if $x \geq 0$, while if $x < 0$ then $|x| = -x$ and $x \geq -t$ so that $-x \leq t$; consequently we have $|x| \leq t$.

(xiii) *Page 5, Proof of Proposition 1.1.9*

If $x, y \in \mathbf{R}$, then (putting $t = |x|$ and $|y|$ in $(*)$ in Note (xii))

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|,$$

so that

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

Using Note (xii) again, we conclude that

$$|x + y| \leq |x| + |y|.$$

This inequality is usually called the *triangle inequality*; the reason for this will become clear during your study of (M3) on **S** page 21 in *Unit 3*.

(xiv) *Page 5, Proof of Corollary 1.1.10*

For any x, y in $\mathbf{R}$ we have that

$$x = (x - y) + y;$$

it follows from the triangle inequality that

$$|x| = |(x - y) + y| \leq |x - y| + |y|,$$

and hence

$$|x| - |y| \leq |x - y|.$$

The same argument with x and y interchanged gives that

$$|y| - |x| \leq |y - x| = |x - y|.$$

Thus, combining the last two inequalities,

$$-|x - y| \leq |x| - |y| \leq |x - y|$$

and so

$$||x| - |y|| \leq |x - y|.$$

Remark. The result of Corollary 1.1.10 is often called *the cunning form of the triangle inequality.*

Remark. Notice the 'trick' above of writing x as $(x - y) + y$. It is a common trick in many proofs in analysis to regroup expressions in a more convenient way. You will see it often in this unit; for example, in SAQ 20(iv). ▲

EXAMPLE 1 For the set $S = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$, find $\sup S$ and $\inf S$.

Since $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, it looks as though the least upper bound of S is 1 since $1 \in S$ and $\frac{1}{n}$ gets smaller as n gets larger. Moreover, since all the elements of S are non-negative and $\frac{1}{n}$ 'approaches' 0 as n gets larger, it looks likely that the greatest lower bound of S is 0. We now verify that both guesses are correct!

First, $\frac{1}{n} \leq 1$ for all $n \in \mathbf{N}$: so that 1 is certainly *an* upper bound for S. But since 1 is itself an element of S, no number less than 1 will serve as an upper bound for S, so that 1 must be the *least* upper bound for S. That is, $\sup S = 1$.

Next, $\frac{1}{n} \geq 0$ for all $n \in \mathbf{N}$; so that 0 is certainly *a* lower bound for S. Note (ix), above, shows that no positive number, ε say, can be a lower bound for S; so that 0 must be the *greatest* lower bound for S. That is, $\inf S = 0$.

Remark. In this case $\sup S$ lies in S, but $\inf S$ does not.

EXAMPLE 2 Let k be a fixed real number, X a non-empty bounded set of real numbers, and $Y = \{kx : x \in X\}$. We shall show that Y is a bounded set; and that, if $k \geq 0$, then

$$\sup Y = k \sup X \quad \text{and} \quad \inf Y = k \inf X;$$

whereas, if $k < 0$, then

$$\sup Y = k \inf X \quad \text{and} \quad \inf Y = k \sup X.$$

(For example, if $k = 3$ and $X = [1, 2]$ then $Y = [3, 6]$; whereas if $k = -3$ and $X = [1, 2]$ then $Y = [-6, -3]$.)

Since S is non-empty and bounded, Axiom 1.1.4 and Proposition 1.1.5 guarantee that

$$a = \inf X \quad \text{and} \quad b = \sup X$$

exist. In particular, $a \leq x \leq b$ for each x in X.

If $k \geq 0$ then $ka \leq kx \leq kb$ if $x \in X$, whilst if $k < 0$ then $kb \leq kx \leq ka$ if $x \in X$. In both cases we see that the set Y of real numbers kx, where $x \in X$, is bounded.

If $k = 0$, then $Y = \{0\}$ and

$$\sup Y = \inf Y = 0 = ka = kb.$$

Next suppose that $k > 0$; we have to prove that $\sup Y = kb$ and $\inf Y = ka$. Now if u is any upper bound for Y then $kx \leq u$ for all x in X. Since $k > 0$ we have $x \leq u/k$ for all x in X, so that u/k is an upper bound for X. Hence $b \leq u/k$, or $u \geq kb$. But we have already seen that kb is an upper bound for Y, so that $\sup Y = kb$. Similarly, if v is a lower bound for Y then v/k is a lower bound for X so that $v/k \leq a$ and so $v \leq ka$. As ka is a lower bound for Y, it follows that $\inf Y = ka$.

Finally, suppose that $k < 0$. We saw above that in this case ka and kb are upper and lower bounds for Y, respectively. If u is any upper bound for Y then $kx \leq u$ for all x in X; so that, since $k < 0$, we have $x \geq u/k$ for all x in X, and so u/k is a lower bound for X. Hence $a \geq u/k$, so that $u \geq ka$. Therefore $\sup Y = ka$. Similarly, if v is a lower bound for Y then v/k is an upper bound for X so that $v/k \geq b$ and so $v \leq kb$. Therefore $\inf Y = kb$.

EXAMPLE 3 (S Exercise 1.5.7, page 16.) First of all we show that if r is any non-zero rational number and t an irrational number, then $r + t$ and rt are irrational numbers. From this we shall deduce that between any two distinct real numbers there is an irrational number (the, so-called, *irrational density property* of $\mathbf{R}$).

(*Remark.* We shall use the facts that the sum, difference, product and quotient of two rational numbers are themselves rational.)

Suppose first that $r + t$ were rational. Then since we can express t in the form

$$t = (r + t) - r,$$

it would follow from the above Remark that t is rational; this contradiction shows that $r + t$ must be irrational.

Suppose next that rt were rational. Since we can express t in the form

$$t = (rt)/r.$$

the Remark would give that t is rational; this contradiction shows that rt must be irrational.

Finally, let $x, y \in \mathbf{R}$ with $x < y$. By the rational density property (S Corollary 1.1.7), there exists a rational number q such that $x < q < y$. Applying the property again, there is a rational number q' such that $q < q' < y$. By the Archimedean property (S Proposition 1.1.6), there exists some n in $\mathbf{N}$ with $n > \sqrt{2}/(q' - q)$ (do not worry about why we chose $\sqrt{2}/(q' - q)$—it is not obvious but it does work!). Since $q' - q > 0$, it follows that $q' - q > \sqrt{2}/n$, or $q' > q + \sqrt{2}/n$. This gives us the relation

$$x < q < q + \sqrt{2}/n < q' < y.$$

Since $\sqrt{2}$ is an irrational number, it follows from the earlier part of the Example that $q + \sqrt{2}/n$ is an irrational number lying between the original real numbers x and y.

SAQ 1 Which of the sets

(a) $S_1 = \{x \in \mathbf{R} : x^2 + x + 1 \geq 0\}$

(b) $S_2 = \{x \in \mathbf{R} : x^2 - 1 < 0\}$

are bounded above and which are bounded below? Where appropriate, find the least upper bound or greatest lower bound.

SAQ 2 Determine (a) a rational number (b) an irrational number, between 1.01 and 1.001.

SAQ 3 Prove that if $x, y \in \mathbf{R}$, then

$$\text{Max}\{x, y\} = \tfrac{1}{2}(x + y + |x - y|)$$

and

$$\text{Min}\{x, y\} = \tfrac{1}{2}(x + y - |x - y|).$$

SAQ 4 (a) Show that the sets

$$S_1 = [0, 1] \cap \mathbf{Q}, \quad S_2 = [0, 1] - \mathbf{Q}$$

both have least upper bound 1 and greatest lower bound 0.

(b) State without proof the least upper bounds and greatest lower bounds of the sets

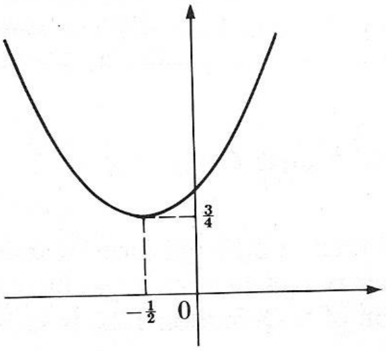
$$S_3 = [a, b] \cap \mathbf{Q}, \quad S_4 = [a, b] - \mathbf{Q}$$

for any real numbers a, b with $a < b$.

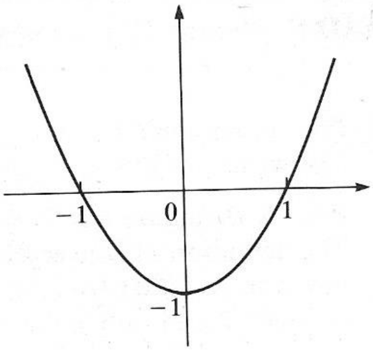
SOLUTION TO SAQ 1 (a) Since

$$x^2 + x + 1 = (x + \tfrac{1}{2})^2 + \tfrac{3}{4} \geq \tfrac{3}{4} > 0 \quad \text{for all } x,$$

it follows that $S_1 = \mathbf{R}$. So S_1 is bounded neither above nor below.



Graph of $y = x^2 + x + 1$



Graph of $y = x^2 - 1$

(b) By considering the sign of $x^2 - 1 = (x - 1)(x + 1)$ in each of the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$ we see that $S_2 = (-1, 1)$. So, in particular, S_2 is both bounded above and bounded below. Since for *any* interval I with end-points a and b, where $a < b$, $\sup I = b$ and $\inf I = a$, it follows that

$$\sup S_2 = 1 \quad \text{and} \quad \inf S_2 = -1. \quad \blacksquare$$

SOLUTION TO SAQ 2

(a) The number 1.005 will do; for it is rational (it equals 1005/1000) and

$$1.001 < 1.005 < 1.01.$$

(b) The number $t = 1 + 0.001 \times \sqrt{2}$ will do. Clearly it lies between 1.001 and 1.01 since $1 < \sqrt{2} < 10$; and since $\sqrt{2}$ is irrational, so are $0.001 \times \sqrt{2}$ and t (where we have been using the results of the first part of Example 3). $\blacksquare$

SOLUTION TO SAQ 3

If $x \le y$ then $|x - y| = y - x$ so that

$$\tfrac{1}{2}(x + y + |x - y|) = y = \text{Max}\,\{x, y\};$$

and if $x > y$ then $|x - y| = x - y$ so that

$$\tfrac{1}{2}(x + y + |x - y|) = x = \text{Max}\,\{x, y\}.$$

The proof of the formula for Min $(x, y\}$ is similar. $\blacksquare$

SOLUTION TO SAQ 4

(a) S_1 consists of all rational numbers in $[0, 1]$. Since 1, 0 both belong to S_1, it is quite reasonable to guess that $\sup S_1 = 1$ and $\inf S_1 = 0$.

1 is certainly *an* upper bound for S_1, and since $1 \in S_1$ no number smaller than 1 can be an upper bound for S_1. Hence $\sup S_1 = 1$.

0 is certainly *a* lower bound for S_1, and since $0 \in S_1$ no number greater than 0 can be a lower bound for S_1. Hence $\inf S_1 = 0$.

(b) S_2 consists of all irrational numbers in $[0, 1]$. Neither 1 nor 0 belongs to S_2, but it still seems likely that $\sup S_2 = 1$ and $\inf S_2 = 0$.

1 is certainly *an* upper bound for S_2. So now suppose that v is an upper bound for S_2 with $v < 1$. Since $1/\sqrt{2}$ lies in S_2, it follows that $1/\sqrt{2} \le v < 1$; consequently, by the irrational density property (Example 3), there is an irrational number x with $1/\sqrt{2} \le v < x < 1$. Hence x is an element of $S_2 = [0, 1] - \mathbf{Q}$ that is greater than v, and so v cannot be an upper bound for S_2. This contradiction completes the proof that $1 = \sup S_2$.

(c) Finally,

$$\sup S_3 = \sup S_4 = b \quad \text{and} \quad \inf S_3 = \inf S_4 = a.$$

(A detailed proof would incorporate arguments similar to those in (a) and (b).) $\blacksquare$

2.2 Real Sequences

In this section we study sequences of real numbers, and the 'ε-N definition' of convergence of a sequence.

READ S Section 1.2 from page 5, omitting Theorem 1.2.7 and Theorem 1.2.10.

NOTES

(i) *Page 6, Example 1.2.1(c)*

The writers of this unit failed to spot the 'simple formula' of **S**!

(ii) *Page 6, Definition 1.2.2*

M203

The definition of convergence in M203 takes a different form. It is shown, however, in M203 *Unit 30*, page 66, that a sequence *converges* in the sense of M203 if and only if the ε-N condition of **S** Definition 1.2.2 is satisfied.

(iii) *Page 6, Definition 1.2.2*
Notice that the condition $|s_n - l| < \varepsilon$ is equivalent to the inequalities $-\varepsilon < s_n - l < \varepsilon$ (since $\varepsilon > 0$), and so to

$$l - \varepsilon < s_n < l + \varepsilon.$$

This should be borne in mind when considering the helpful diagrammatic illustration on **S** page 7 of the definition of convergence.

(iv) *Page 6, Definition 1.2.2*
The rational density property of **R** (**S** Corollary 1.1.7) is frequently exploited in the following way.

Let $a \in \mathbf{R}$. Then for each positive integer n, there exists a rational number s_n such that $a < s_n < a + \frac{1}{n}$. This sequence (s_n) converges to a.

To see this, let $\varepsilon > 0$ be given. It follows from **S**'s Archimedean property of **R** (see Note (ix) of Section 2.1) that there exists a positive integer N such that $1/N < \varepsilon$. Then for all $n \geq N$ we have that

$$|s_n - a| = s_n - a < \frac{1}{n} \leq 1/N < \varepsilon.$$

Therefore (s_n) converges to a according to Definition 1.2.2.

In a similar way we can use the irrational density property of **R** (Example 3 of Section 2.1) to show that corresponding to each $a \in \mathbf{R}$ there exists a sequence of irrational numbers which converges to a. Both facts in this note will be exploited many times in Sections 2.3 and 2.4.

(v) *Page 8, Lemma 1.2.4*
This Lemma, called by **S** a technical detail, is a key observation which enables us to simplify the details of many proofs in our future work. See, for example, Note (ix) below.

(vi) *Page 9, Theorem 1.2.9*
In the proof, **S** shows that every Cauchy sequence is bounded. It follows that every convergent sequence (s_n), being a Cauchy sequence, is bounded.

Notice that Lemma 1.2.4 is used in several places to smooth the argument in the proof of Theorem 1.2.9.

(vii) *Page 9, Theorem 1.2.9*

M203

In M203, the Cauchy convergence criterion is stated in the form: a sequence (s_n) converges if and only if for each $j \in \mathbf{N}$ there is a positive integer N such that $|s_n - s_m| < \frac{1}{2^j}$ for all $n, m \geq N$ (M203 *Unit 30*, page 40).

The (easy) proof that a sequence satisfying this condition is a Cauchy sequence is given in M203 *Unit 30*, page 69.

Cauchy sequences will be considered in a more general setting in *Unit 8*, when we shall see that Theorem 1.2.9 expresses a very important property of **R**.

(viii) *Page 10, lines -7 and -6*
The real point behind this remark in **S** is that the convergence or otherwise of a sequence is unchanged if a *finite* number of terms in the sequence are added, deleted, or changed. For example, all the following sequences converge to 0:

$$\left\{3, 2, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \frac{1}{(n-2)}, \ldots\right\},$$

$$\left\{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots, \frac{1}{(n+1)}, \ldots\right\} \text{ and}$$

$$\left\{0, 2, 0, 0, 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots, \frac{1}{(n-5)}, \ldots\right\}.$$

(ix) *Page 10, Proposition 1.2.11(a) and (b)*
By looking closely at proofs of these we hope to show you how Lemma 1.2.4 can be used in such proofs to save a lot of complicated 'book-keeping'.

(a) Let ε be a given positive real number. Then, by Definition 1.2.2 we know that there exist integers N_1, N_2 such that

$$\left.\begin{array}{ll} |s_n - l| < \varepsilon & \text{for all } n \geq N_1 \\ |t_n - m| < \varepsilon & \text{for all } n \geq N_2. \end{array}\right\} \quad (1)$$

Since, as mentioned in the previous note, we are only interested in s_n and t_n for large n, we may as well suppose that n is greater than or equal to both N_1 and N_2. In other words,

$$|s_n - l| < \varepsilon \quad \text{and} \quad |t_n - m| < \varepsilon \qquad \text{for all } n \geq \text{Max}\,\{N_1, N_2\}.$$

To show that $s_n + t_n$ converges to $l + m$, we must look at the difference $(s_n + t_n) - (l + m)$. So, for all $n \geq \text{Max}\,\{N_1, N_2\}$,

$$\begin{aligned} |(s_n + t_n) - (l + m)| &= |(s_n - l) + (t_n - m)| && \text{(by rearrangement)} \\ &\leq |s_n - l| + |t_n - m| && \text{(by the triangle inequality)} \\ &< \varepsilon + \varepsilon && \text{(for each large } n\text{)} \\ &= 2\varepsilon. && (2) \end{aligned}$$

Lemma 1.2.4, with $K = 2$, shows that $(s_n + t_n)$ converges to $l + m$.

Suppose, however, that we had wanted to avoid using Lemma 1.2.4; that is, we wished to end up, instead of (2), with

$$|(s_n + t_n) - (l + m)| < \varepsilon \quad \text{for all sufficiently large } n. \qquad (3)$$

To do this we should have chosen different N_1 and N_2 in (1) so that $|s_n - l| < \frac{1}{2}\varepsilon$ and $|t_n - m| < \frac{1}{2}\varepsilon$ for n larger than (the new values of) N_1 and N_2. If you follow through the rest of the proof you will see that you do end up with (3) rather than (2).

Of course, there was no way of knowing in advance that we should have worked with $\frac{1}{2}\varepsilon$ in (1) !—so that Lemma 1.2.4 saved us going through the argument twice.

(b) Let $\varepsilon > 0$ be given. Then, as in (a), we know that N_1 and N_2 exist such that (1) holds. To look at the convergence of $(s_n t_n)$ to lm, we clearly have to look at the difference $(s_n t_n) - (lm)$ for large n. Thus, for all $n \geq \text{Max}\,\{N_1, N_2\}$ (chosen so that both inequalities in (1) hold),

$$\begin{aligned} |s_n t_n - lm| &= |s_n(t_n - m) + m(s_n - l)| && \text{(a cunning rearrangement!)} \\ &\leq |s_n|\,|t_n - m| + |m|\,|s_n - l| && \text{(by the triangle inequality)} \\ &\leq |s_n|\varepsilon + |m|\varepsilon. \end{aligned}$$

This upper bound involves n, and so is not of the form (constant) × ε. Therefore we must next use the result in Note (vi) that since (s_n) is convergent, there exists some constant $K > 0$ such that

$$|s_n| < K \quad \text{for all } n.$$

It follows that, for all $n \geq \text{Max}\,\{N_1, N_2\}$, we have

$$\begin{aligned} |s_n t_n - lm| &< K\varepsilon + |m|\varepsilon \\ &= (K + |m|)\varepsilon. \qquad (4) \end{aligned}$$

Lemma 1.2.4, with K replaced by $K + |m|$, shows that $(s_n t_n)$ converges to lm.

Suppose, however, that we had wished to end up with $|s_n t_n - lm| < \varepsilon$ for all sufficiently large n, rather than (4). To do this, we would have to have first recalled the existence of a K with $|s_n| < K$ for all n, and then chosen N_1 and N_2 so that the upper bounds in (1) were $\varepsilon/(K + |m|)$ rather than ε; following the details through, it is easy to verify that this does the trick. But whoever would have guessed that $\varepsilon/(K + |m|)$ would work ?!

The moral is that Lemma 1.2.4 is a very valuable observation—$K\varepsilon$ will do just as well as ε in Definition 1.2.2 to ensure convergence of a sequence. ▲

XAMPLE 4 Let us discuss the convergence of the sequence (s_n) in S Example 1.2.1(a), where $s_n = \dfrac{n}{n+1}$.

If we rewrite the formula for s_n as $s_n = 1 - \dfrac{1}{n+1}$, it is easy to guess what must happen; as n gets larger, $\dfrac{1}{(n+1)}$ 'approaches' 0 and so s_n 'approaches' the value 1. As it is possible for our native intuition to be misled sometimes, we check that this guess is correct using Definition 1.2.2.

The Definition tells us to look at the difference $s_n - 1$; here $|s_n - 1| = \dfrac{1}{(n+1)}$. So if $n \geq 9$, $\dfrac{1}{(n+1)} < \dfrac{1}{10}$, so that $|s_n - 1| < \dfrac{1}{10}$ for all $n \geq 9$. Similarly, $|s_n - 1| < \dfrac{1}{1000}$ for all $n \geq 999$. However, it is not enough to check that the condition of Definition 1.2.2 holds for *some* fixed small values of ε such as 10^{-1} or 10^{-3}; we have to show that the condition holds for *every* positive number ε.

So let ε be a given positive number. By S's Archimedean Axiom, there is an $N \in \mathbf{N}$ with $N > \dfrac{1}{\varepsilon}$ $\left(\dfrac{1}{\varepsilon}\right.$ is chosen simply because it makes the rest of the proof work!$\Big)$. Hence, for all $n \geq N$,

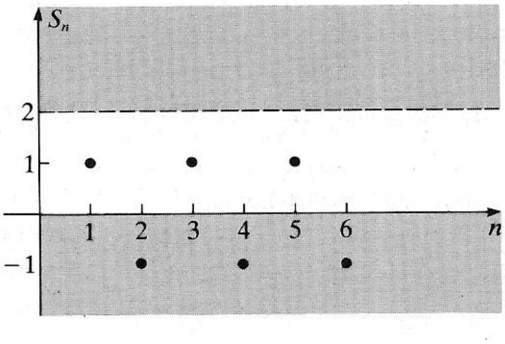
$$|s_n - 1| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon;$$

this shows that the sequence indeed converges to 1.

Remark. This example illustrates the fact that the value we choose for N will depend on that of ε. For example, here, the smaller ε is, the larger N has to be.

However, N is not uniquely determined by ε. For if $|s_n - l| < \varepsilon$ for all $n \geq N$, we still have $|s_n - l| < \varepsilon$ for all $n \geq N'$ for *any* choice we please of an N' that is greater than N. To prove convergence, we do not have to find the best (that is the least) value of N that works—often it is not obvious how to do so. *Any* N that satisfies the required condition '$|s_n - l| < \varepsilon$ for all $n \geq N$' will do.

EXAMPLE 5 Let us show that the sequence (s_n) where $s_n = (-1)^{n+1}$, that is the sequence $\{1, -1, 1, -1, \ldots\}$, does not converge to $+1$. If (s_n) converged to 1 as $n \to \infty$, we would expect that 'eventually' all values of s_n would be 'near to' 1. This is not the case, as half of them are a distance 2 from $+1$; and, in particular, do not lie in a horizontal strip of width 1 about the value 1. Since (as S points out in (2) on page 7) s_n should eventually stay out of the shaded territory if it converged to 1, we see 'intuitively' that (s_n) does not converge to 1. We now prove this carefully, using Definition 1.2.2.



As we often do when asked to prove that something *does not happen*, we use a contradiction argument. Suppose that (s_n) does converge to 1. Then, corresponding to any given $\varepsilon > 0$, there exists an $N \in \mathbf{N}$ such that

$$|s_n - 1| < \varepsilon \quad \text{for all } n \geq N;$$

that is

$$|(-1)^{n+1} - 1| < \varepsilon \quad \text{for all } n \geq N. \qquad (*)$$

$(*)$ has to hold for both even and odd $n \geq N$; the inequality then reads $2 < \varepsilon$, $0 < \varepsilon$ respectively. This suggests that for a contradiction we might agree to choose $\varepsilon = 1$ initially. Our hypothesis then implies the existence of some number N such that

$$2 < 1 \text{ for even } n \geq N, \quad \text{and } 0 < 1 \text{ for odd } n \geq N.$$

The first inequality is absurd! So (s_n) could not have converged to 1 in the first place.

Remark on strategy for showing that a sequence (s_n) does not converge to a given value l. What we have to do is to find a value of ε such that the condition

$$|s_n - l| < \varepsilon \quad \text{for } \textit{all } n \geq N$$

cannot be satisfied for any choice whatever of N.

EXAMPLE 6 To illustrate the usefulness of **S** Theorem 1.2.6, let us consider the sequence (s_n), where

$$s_n = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

(mentioned on **S** page 8).

This sequence is clearly monotonic increasing, as $s_{n+1} - s_n = \dfrac{1}{(n+1)!} > 0$. Theorem 1.2.6 will thus guarantee the convergence of the sequence *if* we can show that the sequence is bounded above. Now, for $r \geq 2$,

$$r! = 1 \times 2 \times \cdots \times r \geq 1 \times 2 \times \cdots \times 2 = 2^{r-1}$$

so that $\dfrac{1}{r!} \leq \dfrac{1}{2^{r-1}}$, and hence

$$\begin{aligned} s_n &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &= 1 + 2\left(1 - \frac{1}{2^n}\right) \\ &< 3. \end{aligned}$$

Since $1 < s_n < 3$, the sequence is indeed bounded as required—and so convergent.

Remark. Theorem 1.2.6 does *not* determine the value of the limit of (s_n). Of course, in Example 6 we know that the limit is $e = \exp(1)$.

Now play the first part of the tape for Unit 2

SAQ 5 Show that the sequence (s_n), where $s_n = n^{-1/2}$, converges to 0.

SAQ 6 Show that the sequence (s_n), where $s_n = (-1)^{n+1}$, does not converge. (You saw in Example 5 that it certainly does not converge to 1.)

SAQ 7 (a) Give an example of a sequence (s_n) which does not converge, although the sequence $(|s_n|)$ does converge. (No proofs needed.)

(b) Prove that, if the sequence $(|s_n|)$ converges to 0, then so does the sequence (s_n).

SAQ 8 Does the following sequence converge to 0?

$$s_n = \begin{cases} 10^{-10}, & \text{if } n \text{ is a multiple of 100,} \\ \dfrac{1}{n}, & \text{if } n \text{ is not a multiple of 100.} \end{cases}$$

Prove your claim *carefully*, using an 'ε-N argument'.

SAQ 9 (a) Does the sequence (s_n), where $s_n = x^n$ and $0 < x < 1$, converge? If it converges, state its limit.

(b) Does the sequence (s_n), where

$$s_n = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3},$$

converge? [Hint: Use the fact that $n < 2^n$ for each $n \in \mathbf{N}$.]

SAQ 10 Prove Proposition 1.2.11(c). (You may assume, without proof, that constants K and N exist such that $|t_n| \geq K > 0$ for all $n \geq N$.)

SOLUTION TO SAQ 5 First of all, let ε be any given positive number.

We have to examine the difference $|s_n - 0| = n^{-1/2}$. Since $n^{-1/2}$ is decreasing, this will be less than ε for all $n \geq N$ if $N^{-1/2} < \varepsilon$; that is, if $N^{-1} < \varepsilon^2$ or equivalently $\frac{1}{\varepsilon^2} < N$. This suggests just how to provide the detailed proof!

By the Archimedean axiom, there exists an $N \in \mathbf{N}$ such that $N > \frac{1}{\varepsilon^2}$. Then, for all $n \geq N$, we have

$$|s_n - 0| = n^{-1/2} \leq N^{-1/2} < \varepsilon.$$

This shows that (s_n) converges to 0, as desired. ■

SOLUTION TO SAQ 6 Following the advice in Example 5, we use a contradiction argument.

Suppose that a real number l exists such that (s_n) converges to the limit l. Then, corresponding to any given $\varepsilon > 0$, there exists an $N \in \mathbf{N}$ such that

$$|s_n - l| < \varepsilon \quad \text{for all } n \geq N;$$

that is, $|(-1)^{n+1} - l| < \varepsilon$ for all $n \geq N$.

Whatever N is, there are both even and odd n that must satisfy this inequality ($N + 1$ and $N + 2$, for instance!) hence we must have

$$|1 + l| < \varepsilon \quad \text{and} \quad |1 - l| < \varepsilon.$$

n has now disappeared; and, whatever l is, both inequalities have to be satisfied for every $\varepsilon > 0$. Whenever you see expressions like $1 + l$, $1 - l$ you should immediately think of applying the triangle inequality to one of the expressions

$$(1 + l) + (1 - l) \quad \text{or} \quad (1 + l) - (1 - l).$$

Here we use the first; so

$$\begin{aligned} 2 &= |(1 + l) + (1 - l)| \\ &\leq |1 + l| + |1 - l| < 2\varepsilon. \end{aligned}$$

So whatever $\varepsilon > 0$ is, we have $2 < 2\varepsilon$; so, if at the beginning we choose $\varepsilon = \frac{1}{2}$ so that $2 < 1$, we get the desired contradiction. ■

Remark. Write out the same solution in a less chatty form than ours!

SOLUTION TO SAQ 7 (a) Take $s_n = (-1)^{n+1}$. We saw in SAQ 6 that (s_n) does not converge; and it is quite easy to check that, since $|s_n| = 1$ for each n, $(|s_n|)$ converges (to 1).

(b) Let $\varepsilon > 0$ be given. Then, since $(|s_n|)$ converges to 0, there is an $N \in \mathbf{N}$ such that

$$||s_n| - 0| < \varepsilon \quad \text{for all } n \geq N;$$

we can rewrite this as

$$|s_n| < \varepsilon \quad \text{for all } n \geq N,$$

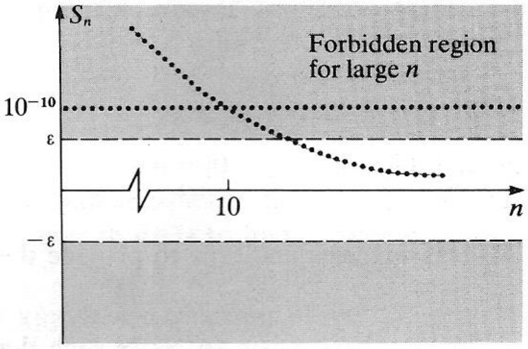
or

$$|s_n - 0| < \varepsilon \quad \text{for all } n \geq N.$$

This is precisely the condition for (s_n) to converge to 0. ■

SOLUTION TO SAQ 8

Since we can easily show that the sequence $\left(\frac{1}{n}\right)$ converges to 0, and 99% of the terms of the sequence (s_n) are the corresponding terms of the sequence $\left(\frac{1}{n}\right)$, we might be led to the conclusion that $s_n \to 0$ as $n \to \infty$. But wait a minute!—what about the other terms? Every hundredth term (of which there are infinitely many) is a distance 10^{-10} away from 0 (see the sketch of s_n against n). So it looks as though the sequence does *not* converge to 0.



So suppose that we take $\varepsilon = \frac{1}{2} \times 10^{-10}$, and suppose that $s_n \to 0$. Then there exists some $N \in \mathbf{N}$ such that

$$|s_n - 0| < \tfrac{1}{2} \times 10^{-10} \quad \text{for all } n \geq N;$$

in particular, when $n = 10^{10} \times N$, so that $s_n = 10^{-10}$, we have that

$$10^{-10} < \tfrac{1}{2} \times 10^{-10}.$$

This is absurd! Therefore this contradiction shows that the sequence does not converge to 0. ■

Remark. We could in fact show that (s_n) does not converge to *any* limit, using the same method as in the Solution to SAQ 6.

SOLUTION TO SAQ 9

(a) Since $x > 0$, $x^n > 0$ for each n; and since $x < 1$, $x^n < 1$ for each n. Thus (s_n) takes values only in $(0, 1)$, and so is bounded.

Also, (s_n) is monotonic decreasing, since

$$\begin{aligned} s_{n+1} = x^{n+1} &= x \cdot x^n = xs_n \\ &< s_n \qquad \text{since } 0 < x < 1. \end{aligned}$$

Theorem 1.2.6 then guarantees that (s_n) converges.

In fact, $s_n \to 0$ as $n \to \infty$.

(b) Since $s_{n+1} - s_n = \dfrac{1}{(n+1)^3} > 0$, the sequence is monotonic increasing. Next, since $n < 2^n$, $n^3 < 2^{3n}$; it follows that

$$\begin{aligned} s_n &< 1 + \frac{1}{2^3} + \frac{1}{2^6} + \cdots + \frac{1}{2^{3n}} \\ &< \sum_{n=0}^{\infty} \left(\frac{1}{2^3}\right)^n = \frac{1}{1 - \frac{1}{8}} = \frac{8}{7}. \end{aligned}$$

And since $s_n \geq 1$, for each n we have that $s_n \in [1, \frac{8}{7}]$. Since the hypotheses of Theorem 1.2.6 are satisfied, (s_n) converges. ■

Remark. Although we have shown that $s_n \to l$ as $n \to \infty$ for some real number l, the efforts of mathematicians over the centuries have revealed very little about the value of this l. In fact, it was only in 1978 that it was proved that l is irrational!

SOLUTION TO SAQ 10

Let any $\varepsilon > 0$ be given. We know that we have to look at the difference $\left|\frac{1}{t_n} - \frac{1}{m}\right|$; if we write this in the form $\dfrac{|t_n - m|}{|t_n| \cdot |m|}$ and remember that we are looking for an upper bound for this expression, this gives us a guide how to proceed.

Since $t_n \to m$, there exists an $N \in \mathbf{N}$ such that $|t_n - m| < \varepsilon$ for all $n \geq N$. Hence, for all $n \geq N$,

$$\begin{aligned}\left|\frac{1}{t_n} - \frac{1}{m}\right| &= \frac{|t_n - m|}{|t_n| \cdot |m|} \\ &< \frac{\varepsilon}{|t_n||m|} \qquad \text{(as described above)} \\ &< \left(\frac{1}{K|m|}\right)\varepsilon \qquad \text{(using the assumption).}\end{aligned}$$

Lemma 1.2.4, with K replaced by $\dfrac{1}{K|m|}$, shows that $\dfrac{1}{t_n} \to \dfrac{1}{m}$ as $n \to \infty$. ■

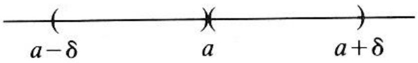
2.3 Limits of Functions

READ S all of Section 1.3 from page 11.

NOTES *M203*

(i) *Page 11, Definition 1.3.1*
In M203 *Unit 30*, pages 58 and 62, it is shown that $\lim_{x \to a} f(x) = l$ in the M203-sense if and only if the 'ε-δ condition' formulated in Definition 1.3.1 is satisfied.

(ii) *Page 11, Definition 1.3.1*
Note that a real number x satisfies $0 < |x - a| < \delta$ if and only if $|x - a| < \delta$ and $x \neq a$; that is, if and only if $x \in (a - \delta, a + \delta)$ and $x \neq a$. Thus $0 < |x - a| < \delta$ if and only if $x \in (a - \delta, a) \cup (a, a + \delta)$.



(iii) *Page 11, Definition 1.3.1*
The following analogue of Lemma 1.2.4 for convergence of sequences can be (quickly) proved for limits of functions.

Suppose that given $\varepsilon > 0$ there exists a (real number) $\delta > 0$ such that $|f(x) - l| < K\varepsilon$ for all x satisfying $0 < |x - a| < \delta$, where K is a (positive) real number independent of ε and δ. Then $f(x)$ tends to the limit l as x tends to a.

(iv) *Page 11, Definition 1.3.1*
In addition to a strategy for checking that $f(x)$ tends to a limit l, we need a strategy for checking that $f(x)$ does *not* tend to a limit l. (cf. Remark after Example 5 in Section 2.2.)

If $f(x) \to l$ as $x \to a$, then, for each given positive number ε, there has to exist a δ with a particular property. The opposite of this is that: $f(x)$ does *not* tend to l as $x \to a$ if we can find an $\varepsilon > 0$ such that *no* corresponding δ exists with that property. The existence of just *one* such 'awkward' ε is enough!

So let us denote a particular awkward ε by ε_0. Then $f(x)$ does not tend to the limit l as $x \to a$ if: there exists some $\varepsilon_0 > 0$ such that *whatever* $\delta > 0$ we choose it is *not* true that

$$|f(x) - l| < \varepsilon_0 \quad \text{for } all\ x \text{ with } 0 < |x - a| < \delta.$$

It is easier to handle a statement that such-and-such *is* true rather than *is not* true; so we rephrase the above criterion as: there exists some ε_0 such that: for each $\delta > 0$ there exists *some* x such that

$$|f(x) - l| \geq \varepsilon_0 \text{ and } 0 < |x - a| < \delta.$$

Remark. You will find it valuable to read Example 8 below at this point, before continuing with the Notes.

(v) *Page 12, lines 1 and 2*
The *left-hand limit* $\lim_{x \to b-} f(x)$ equals l if given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \text{for all } x \in (b - \delta, b).$$

(Here, to avoid unnecessary fuss, we have implicitly assumed that f is defined on some set containing the open interval (a, b), and that $a \leq b - \delta < b$.)

The idea of left- and right-hand limits is of importance when we consider real-valued functions whose domain is a bounded interval in **R**, so that we can talk about the behaviour at the end-points of the interval.

(vi) *Page 12, Proof of Lemma 1.3.5*
(To show that (i) and (ii) are equivalent we have to show that (i) implies (ii) and that (ii) implies (i).)

So suppose first that $\lim_{x \to a} f(x) = l$, and let (x_n) be a sequence which converges to a such that for each n, $x_n \neq a$; we wish to show that $f(x_n)$ converges to l.

Let any $\varepsilon > 0$ be given. Then there exists, by Definition 1.3.1, some $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \text{for all } x \text{ satisfying } 0 < |x - a| < \delta. \qquad (*)$$

Now since the sequence (x_n) converges to a, there exists, by Definition 1.2.2, an integer N such that $|x_n - a| < \delta$ for every $n \geq N$; and, since $x_n \neq a$, it follows that $0 < |x_n - a| < \delta$ for every $n \geq N$. Since x_n then classes as one of the points x that satisfy $(*)$, we can conclude that, for $n \geq N$,

$$|f(x_n) - l| < \varepsilon;$$

this shows that $(f(x_n))$ converges to l, as required.

To prove the opposite assertion, that (ii) implies (i), we use a contradiction argument. Suppose that (ii) holds, but that $f(x)$ does *not* tend to l as x tends to a. Then there exists some number $\varepsilon_0 > 0$ with the following property: for every $\delta > 0$ there exists some x such that

$$|f(x) - l| \geq \varepsilon_0 \quad \text{and} \quad 0 < |x - a| < \delta$$

(see Note (iv)). Taking $\delta = \dfrac{1}{n}$ for each $n \in \mathbf{N}$ in turn, we see that corresponding to each $n \in \mathbf{N}$ there exists some number x_n such that

$$|f(x_n) - l| \geq \varepsilon_0 \quad \text{and} \quad 0 < |x_n - a| < \frac{1}{n}.$$

From the last inequality, it is clear that (x_n) is a sequence which converges to a and that, for each n, $x_n \neq a$. But since $|f(x_n) - l| \geq \varepsilon_0 > 0$ for each n, the sequence $(f(x_n))$ cannot converge to l. Hence (ii) is false. This contradicts our original assumption; so that, if (ii) holds, then so does (i)—in other words, (ii) implies (i).

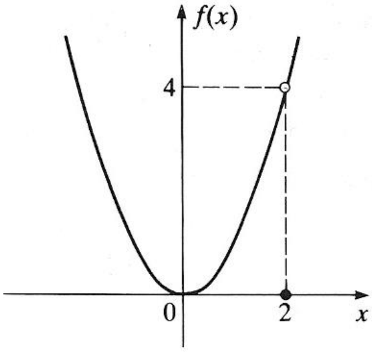
Remark. We shall find a use for this Lemma in Example 10 in the next section, and in SAQs in this and the next section. (In practice it is often valuable to use (ii) as a test to see whether it is actually likely that a function tends to a limit.)

(vii) *Page 12, line 2 after Lemma 1.3.5*
M203 Proofs of the sum and product rules for limits of functions (the analogue of S Proposition 1.2.11) are given in M203 *Unit 30*, pages 62–4, using the ε-δ definition of limit. ▲

EXAMPLE 7 (On the use of S Definition 1.3.1.)

Let us show that the function

$$f: x \longmapsto \begin{cases} x^2, & \text{if } x \neq 2, \\ 0, & \text{if } x = 2, \end{cases}$$



has the limit 4 as x tends to 2.

From the definition, we know that we are supposed to show that for x near to 2, the following difference is small:

$$f(x) - 4 = x^2 - 2^2 = (x-2)(x+2).$$

For x near to 2, the bracket $(x + 2)$ is more-or-less just the number 4, and so the difference is approximately $4(x-2)$; so if $0 < |x-2| < \delta$, $|f(x) - 4|$ is 'less than approximately' 4δ. So if we choose a δ approximately equal to $\frac{1}{4}\varepsilon$, we will have $|f(x) - 4| < \varepsilon$. We make this precise as follows.

Let $\varepsilon > 0$ be given. As we are interested only in x near 2, consider only x that satisfy the condition $x \in (1,3)$, $x \neq 2$; for these, $x + 2 \in (3,5)$, so that $|x+2| < 5$.

JOKE (5 is approximately equal to 4(!).) So if we choose $\delta = \text{Min}\{\frac{1}{5}\varepsilon, 1\}$ and consider only x with $0 < |x-2| < \delta$, it follows that for these x we have both

$$|x-2| < \tfrac{1}{5}\varepsilon \quad \text{and} \quad |x+2| < 5.$$

Consequently,

$$|f(x) - 4| = |x-2| \cdot |x+2| < \varepsilon$$

whenever $0 < |x-2| < \delta$. Hence $f(x) \to 4$ as $x \to 2$.

Remark. We could also have proved this result using the observation in Note (iii) and choosing $\delta = \text{Min}\{\varepsilon, 1\}$; this would result in the final statement

$$|f(x) - 4| < 5\varepsilon \quad \text{whenever } 0 < |x-2| < \delta.$$

For this particular f, then, Note (iii) wouldn't have saved us very much!

Remark. Note that although f tends to the limit 4 as x tends to 2, $f(2)$ does not actually equal 4. This is the whole point of the inequality $0 < |x-a|$ which appears in Definition 1.3.1.

EXAMPLE 8 (On the use of S Definition 1.3.3.)

We show that the function $g: \mathbf{R} - \{0\} \to \mathbf{R}$ defined by

$$g(x) = \begin{cases} 1 + x, & \text{if } x > 0, \\ -1 + x, & \text{if } x < 0, \end{cases}$$

does not tend to a limit as x tends to 0, although g has both left- and right-hand limits at 0.

For the limit at 0, we use the strategy outlined in Note (iv) above. Suppose that some real number l existed such that $\lim_{x \to 0} g(x) = l$. Then, corresponding to any given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|g(x) - l| < \varepsilon \quad \text{whenever } 0 < |x - 0| < \delta.$$

For such x, $g(x)$ must belong to the interval $(l - \varepsilon, l + \varepsilon)$ of length 2ε; if ε is less than $\frac{1}{2}$, so that the interval has length less than 1, it cannot contain values near both ± 1, although, near 0, g is taking values near both ± 1.

This suggests that if we choose an ε less than $\frac{1}{2}$ we should be able to get a contradiction. Let us now write out a formal argument.

Take $\varepsilon = \frac{1}{4}$. Then some $\delta > 0$ exists such that

$$|g(x) - l| < \tfrac{1}{4} \quad \text{whenever } 0 < |x| < \delta.$$

Then, taking $x = \frac{1}{2}\delta$, $x = -\frac{1}{2}\delta$ in turn, so that $g(\frac{1}{2}\delta) = 1 + \frac{1}{2}\delta$, $g(-\frac{1}{2}\delta) = -1 - \frac{1}{2}\delta$ respectively, the following two inequalities must both be satisfied:

$$|1 + \tfrac{1}{2}\delta - l| < \tfrac{1}{4} \quad \text{and} \quad |1 + \tfrac{1}{2}\delta + l| < \tfrac{1}{4}.$$

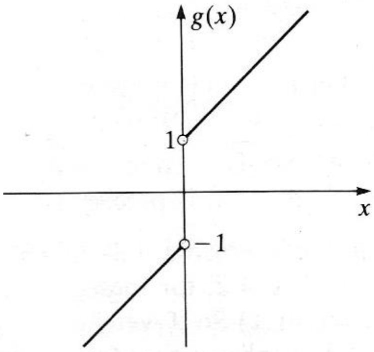
Using the triangle inequality (as we did in a similar situation in the Solution to SAQ 6),

$$|(1 + \tfrac{1}{2}\delta - l) + (1 + \tfrac{1}{2}\delta + l)| \leq |1 + \tfrac{1}{2}\delta - l| + |1 + \tfrac{1}{2}\delta - l|,$$

so that

$$2 + \delta < \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

Since $\delta > 0$, this is a contradiction; consequently $\lim_{x\to 0} g(x)$ does not exist.



From the diagram, it seems reasonable to suppose that g has a right-hand limit 1 at 0; let us prove this carefully. Let any $\varepsilon > 0$ be given. Since, for $x > 0$,

$$g(x) - 1 = (1 + x) - 1 = x,$$

let us choose δ to have the value ε. Then, for all x with $0 < x < \delta(=\varepsilon)$, it follows that

$$|g(x) - 1| = x < \varepsilon;$$

hence $\lim_{x\to 0+} g(x) = 1$.

In a similar way, we can show that g has a left-hand limit -1 at 0. Let $\varepsilon > 0$ be given; choose $\delta = \varepsilon$. Then, for all x with $(-\varepsilon=) -\delta < x < 0$, it follows that

$$\begin{aligned} |g(x) - (-1)| &= |(-1 - x) - (-1)| \\ &= |x| < \varepsilon; \end{aligned}$$

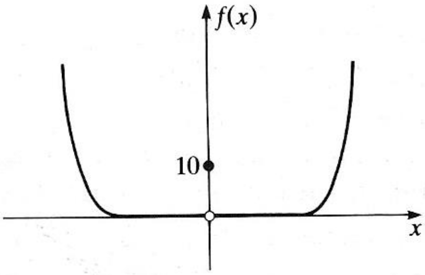
so that, by Note (v), $\lim_{x\to 0-} g(x) = -1$.

Remark. This shows that g can have left- and right-hand limits at a point without having a limit at the point.

SAQ 11 (a) For the function $f:\mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$f(x) = \begin{cases} x^{10}, & \text{if } x \neq 0, \\ 10, & \text{if } x = 0, \end{cases}$$

determine $\lim_{x\to 0} f(x)$. Prove your assertion.



(b) For the function $g:\mathbf{R} - \{0\} \longrightarrow \mathbf{R}$ defined by

$$g(x) = x^{10},$$

determine whether or not $\lim_{x\to 0} g(x)$ exists.

SAQ 12 Let $f:\mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that $\lim\limits_{x \to a} f(x)$ does not exist for any $a \in \mathbf{R}$.

SAQ 13 Let $f:\mathbf{R}_+ \longrightarrow \mathbf{R}$ be given by

$$f(x) = x^{1/2} \quad (x > 0).$$

Prove that (i) $\lim\limits_{x \to a} f(x) = a^{1/2}$ if $a > 0$ and (ii) $\lim\limits_{x \to 0+} f(x) = 0$.

[Hint: $(x^{1/2} - a^{1/2})(x^{1/2} + a^{1/2}) = x - a$ when $x, a \geq 0$.]

SAQ 14 Let $f:\mathbf{R} \longrightarrow \mathbf{R}$ be a function, and let $a \in \mathbf{R}$.

Which of the following statements, if any, are equivalent to the statement that $\lim\limits_{x \to a} f(x)$ exists and is equal to l?

(i) For each given $\delta > 0$, there exists $\varepsilon > 0$ such that $|f(x) - l| < \varepsilon$ for every x such that $0 < |x - a| < \delta$.

(ii) For each given $\delta > 0$, there exists $\varepsilon > 0$ such that $|f(x) - l| < \delta$ for every x such that $0 < |x - a| < \varepsilon$.

(iii) For each given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ implies that $0 < |x - a| < \delta$.

SOLUTION TO SAQ 11

(a) Let any $\varepsilon > 0$ be given. From the diagram it seems likely that the limit is 0; so let us prove it carefully. We know that we have to examine the difference $|f(x) - 0| = x^{10}$ near 0.

If we choose (out of thin air?—not quite; the previous line suggests why!) $\delta = \varepsilon^{1/10} > 0$, then whenever $0 < |x - 0| < \delta$, it follows that

$$\begin{aligned} |f(x) - 0| &= x^{10} \\ &< \delta^{10} = \varepsilon. \end{aligned}$$

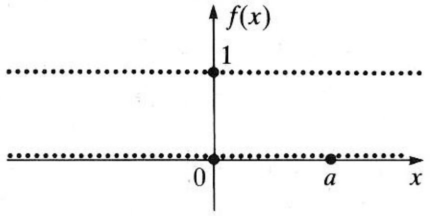
In other words, $\lim\limits_{x \to 0} f(x) = 0$.

(b) *Exactly* the same argument, with f replaced by g, works!—and shows that $\lim\limits_{x \to 0} g(x)$ exists and equals 0. ■

Remark. This shows that the value of a function at a point (or indeed even whether it is defined at the point) makes no difference to the value of its limit there.

SOLUTION TO SAQ 12

In view of the rational density property and the irrational density property (Section 2.1), it is a little difficult to sketch the graph of $f(x)$! We have done our best!



Given any point a and any $\delta > 0$, any set $\{x : 0 < |x - a| < \delta\}$ will contain both rational x and irrational x, and so points where $f(x)$ is 1 and 0 respectively. Whatever value l that $\lim\limits_{x \to a} f(x)$ might be, no interval $(l - \varepsilon, l + \varepsilon)$ with mid-point l can contain both 1 and 0 if ε is chosen to be less than $\frac{1}{2}$. This motivates the following 'proof by contradiction', following the strategy of Note (iv).

Choose $\varepsilon = \frac{1}{4}$, and suppose that $\lim\limits_{x\to a} f(x)$ exists and equals l. Then there exists some $\delta > 0$ such that, whenever $0 < |x - a| < \delta$, then

$$|f(x) - l| < \tfrac{1}{4}.$$

Choosing $x \in (a - \delta, a) \cup (a, a + \delta)$ to be rational and irrational in turn, it follows that

$$|1 - l| < \tfrac{1}{4} \quad \text{and} \quad |l| < \tfrac{1}{4};$$

in other words, $l \in (\frac{3}{4}, \frac{5}{4})$ *and* $l \in (-\frac{1}{4}, \frac{1}{4})$, a contradiction. Thus $f(x)$ has no limit at a.

Alternative Solution (using **S** Lemma 1.3.5)
Let (x_n) be a sequence of rationals which converge to a, and (y_n) a sequence of irrationals converging to a, $x_n \neq a$, $y_n \neq a$. Since $f(x_n) = 1$ for all n, it is easy to check that the sequence $(f(x_n))$ converges to 1; similarly, the sequence $(f(y_n))$ converges to 0. Thus no l exists such that (ii) of Lemma 1.3.5 is satisfied for *every* sequence converging to a, and so the limit $\lim\limits_{x\to a} f(x)$ does not exist. ■

Remark. Appealing to Lemma 1.3.5 is often the most efficient way to show that a limit does *not* exist, but to Definition 1.3.1 the most efficient way to show that it does.

SOLUTION TO SAQ 13

(i) We know that we have to examine the difference

$$|f(x) - a^{1/2}| = |x^{1/2} - a^{1/2}|$$
$$= \frac{|x - a|}{x^{1/2} + a^{1/2}} \qquad \text{(denominator non-zero).}$$

If x is near to a, this ratio is approximately $\dfrac{1}{2a^{1/2}}|x - a|$ and is certainly $\leq \dfrac{1}{a^{1/2}}|x - a|$. It is now clear how to proceed!

Let $\varepsilon > 0$ be given. Taking $\delta = \varepsilon$ (chosen because it works well!), we see that if $0 < |x - a| < \delta (= \varepsilon)$, then

$$|f(x) - a^{1/2}| = \frac{|x - a|}{x^{1/2} + a^{1/2}}$$
$$< \frac{|x - a|}{a^{1/2}} = a^{-1/2}\varepsilon.$$

In the light of Note (iii), with $K = a^{-1/2}$, this shows that $\lim\limits_{x\to a} f(x) = a^{1/2}$.

(ii) (Here, for x near to 0, $|f(x) - 0| = x^{1/2} < \varepsilon$ if $x < \varepsilon^2$. So choosing $\delta = \varepsilon^2$ may well work!)

Let $\varepsilon > 0$ be given, and let $\delta = \varepsilon^2$. Then, for all x such that $0 < x < \delta$, it follows that

$$|f(x) - 0| = x^{1/2}$$
$$< \delta^{1/2} = \varepsilon.$$

This proves that $\lim\limits_{x\to 0+} f(x)$ exists and equals 0. ■

SOLUTION TO SAQ 14

(i) This is not equivalent to Definition 1.3.1.

The function f of SAQ 12 will do as a counter-example. For condition (i) is satisfied for each $a \in \mathbf{R}$, with $l = 0$, by taking $\varepsilon = 2$ for each $\delta > 0$. But SAQ 12 shows that $\lim\limits_{x\to a} f(x)$ does not exist.

(ii) This is precisely Definition 1.3.1—the interchange of the symbols ε and δ is of no significance (it's just a mean trick on our part!).

(iii) This is not equivalent to Definition 1.3.1.
For example, let $a = l = 0$ and let $f:\mathbf{R} \longrightarrow \mathbf{R}$ be the constant function $f:x \longmapsto 0$. Then certainly $\lim\limits_{x \to 0} f(x) = 0$; and also $|f(x) - l| = 0$ for every x, so that for each $\varepsilon > 0$ there is no δ such that

$$|f(x) - l| < \varepsilon \quad \text{implies that} \quad 0 < |x - 0| < \delta,$$

since x can be arbitrarily large. ■

2.4 Continuity

One aim of this option is to study continuity of functions in a very general setting. In this section we look at the continuity of functions with domain and codomain **R**.

READ S all of Section 1.4 from page 12.

NOTES

M203

(i) *Page 13, Definition 1.4.3*
Let us recall the definition of the *limit* of a function at a finite point, given in M203 *Unit 30*, page 28. Let f be a function defined except at a, and let l be a real number. If the function g defined by putting $g(a) = l$ and $g(x) = f(x)$ for every $x \neq a$ is continuous at a, then we say that $\lim\limits_{x \to a} f(x)$ exists and equals l.

(This process of assigning a value at a is called the process of *continuous extension to a*.) Thus in M203 the 'limit as $x \to a$' is defined in terms of continuity at a, whilst in **S** Definition 1.4.3, continuity at a is defined in terms of the limit as x tends to a.

There is, however, no conflict in these two approaches. Indeed, in M203 *Unit 30*, page 58, it is shown that a function f is continuous at a if and only if the ε-δ condition formulated in **S** Definition 1.4.3′ is satisfied.

(ii) *Page 13, Definition 1.4.3*
S Lemma 1.3.5 can be reformulated to provide a useful criterion for continuity: a function $f:\mathbf{R} \longrightarrow \mathbf{R}$ is continuous at $a \in \mathbf{R}$ if and only if, for each sequence (x_n) which converges to a, the sequence $(f(x_n))$ converges to $f(a)$.

(iii) *Page 14, Definition 1.4.3′*
The following analogue of Lemma 1.2.4 and of Note (iii) of the previous section is easy to verify and very useful in proving that a function *is* continuous:

A function $f:\mathbf{R} \longrightarrow \mathbf{R}$ is continuous at a if given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < K\varepsilon \quad \textit{for any } x \textit{ satisfying } |x - a| < \delta,$$

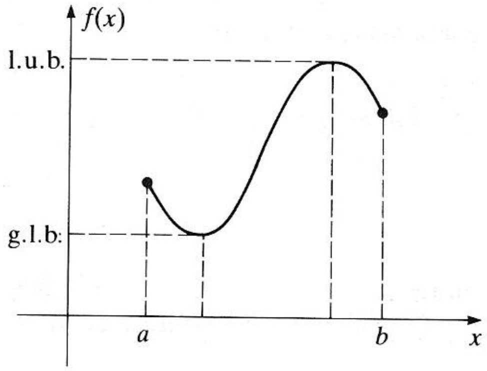
where K is a (positive) real number independent of ε and x.

(iv) *Page 15, line 3*
We say that a function $f:[a, b] \longrightarrow \mathbf{R}$ is continuous on the closed interval $[a, b]$ if

(i) it is continuous at each point of the open interval (a, b), and

(ii) it is continuous from the right at a and from the left at b.

For future use in the option, we mention **S** Proposition 5.1.1: a continuous function $f:[a,b]\longrightarrow\mathbf{R}$ is bounded on $[a,b]$ (and indeed attains its least upper bound and greatest lower bound on $[a,b]$).



(v) *Page 15, Theorem 1.4.5*

M203 It is established in M203 *Unit 10*, page 37, that each continuous function $f:\mathbf{R}\longrightarrow\mathbf{R}$ has the I.V.P. We shall see in *Unit* 7 that this is a consequence of a property of **R** called *connectedness.* ▲

EXAMPLE 9 (On the use of Definition 1.4.3′.)

We prove that the function $f:\mathbf{R}\longrightarrow\mathbf{R}$ defined by $f:x\longmapsto x^2$ is continuous at each point $a\in\mathbf{R}$.

From the Definition we know that we should consider the difference between $f(x)$ and $f(a)$:

$$|f(x)-f(a)|=|x^2-a^2|.$$

The first step is to notice that the right-hand side can be factorized, so that

$$|f(x)-f(a)|=|x-a|\cdot|x+a|; \qquad (1)$$

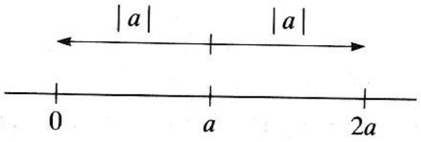
this is obviously a useful formula since it contains both the expressions $|f(x)-f(a)|$ and $|x-a|$ in the Definition. In order to exploit this in detail, it turns out that for convenience we should deal with the cases $a=0$, $a\neq 0$ separately.

When $a=0$, we have seen in (1) that

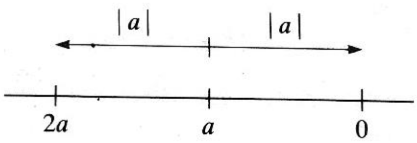
$$|f(x)-f(0)|=|x|\cdot|x|=|x|^2.$$

Hence, for given $\varepsilon>0$, let us choose $\delta=\varepsilon^{1/2}$. It follows that if $|x|<\delta$ then $|x|^2<\varepsilon$, and so $|f(x)-f(0)|<\varepsilon$; thus f is continuous at 0.

When $a\neq 0$, the argument has to be rather different. Loosely, we are trying to show that if $|x-a|$ is small then so is $|f(x)-f(a)|$, or that if x is near to a then $f(x)$ is near to $f(a)$. We are interested only in the behaviour of f near a; there the term $|x+a|$ in (1) is more-or-less just $|2a|$, and so gives a (nearly) constant multiple of $|x-a|$ in (1). Pinning down what we mean by 'near', we argue as follows.



The case $a>0$



The case $a<0$

If $|x-a|<|a|$, then

$$\begin{aligned}|x+a|&=|(x-a)+2a|\\&\leq|x-a|+2|a| \qquad \text{(by the triangle inequality)}\\&<3|a|,\end{aligned}$$

so that

$$|f(x)-f(a)|<(3|a|)\cdot|x-a|. \qquad (2)$$

To make $|f(x) - f(a)| < \varepsilon$, it is then enough to have firstly that $|x - a| < |a|$ (so that (2) holds) and secondly that $3|a||x - a| < \varepsilon$, that is that $|x - a| < \varepsilon/(3|a|)$.

Hence, for given $\varepsilon > 0$, we choose $\delta = \text{Min}\,\{|a|, \varepsilon/(3|a|)\}$. The above argument then shows that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$, so that in this case ($a \neq 0$) too, f is continuous at a.

So f is continuous (on **R**).

EXAMPLE 10 We now show how **S** Lemma 1.3.5 can be used to show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \sin\left(\dfrac{1}{x}\right), & \text{if } x > 0, \end{cases}$$

is not continuous at 0. (See **S** Example 1.4.2, page 13, and **S** Figure 1.3, page 14.) As you might expect by now, to show that f is *not* continuous, we use a contradiction argument.

Suppose that f were continuous at 0. Then by Definition 1.4.3, $\lim_{x \to 0} f(x)$ exists and equals $f(0)$. It follows from Lemma 1.3.5, that for any sequence (x_n) that converges to 0, $x_n \neq 0$, we have that $(f(x_n))$ converges to $f(0)(= 0)$. But if we take $x_n = \dfrac{2}{(4n + 1)\pi}$, then $x_n \neq 0$, (x_n) converges to 0, and

$$f(x_n) = \sin\,(4n + 1)\frac{\pi}{2} = 1,$$

so that $(f(x_n))$ does not converge to 0. This contradiction shows that f is not continuous at 0.

Remark. The method we used here is probably the most common way of showing that a function is not continuous.

Now play the second part of the tape for Unit 2

SAQ 15 Show that the function $f(x) = x^3$ is continuous at 1.

SAQ 16 Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational}, \\ 0, & \text{if } x \text{ is irrational}. \end{cases}$$

(i) Show that f is continuous at 0.

(ii) If $a \neq 0$, show that f is not continuous at a.

SAQ 17 Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ and $g: \mathbf{R} \longrightarrow \mathbf{R}$ be continuous functions such that $f(x) = g(x)$ if x is a rational number. Prove that $f = g$.

[Hint: Use the rational density property.]

***SAQ 18** For each real number x, let $[x]$ denote the integer part of x; that is, the largest integer not exceeding x. Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by $f(x) = [x]$. Prove that

(i) if a is not an integer, f is continuous at a; and

(ii) if a is an integer, f is not continuous at a.

SAQ 19 Prove that, if $f: \mathbf{R} \longrightarrow \mathbf{R}$ is continuous at a and $f(a) > 0$, then there exists some $\delta > 0$ such that

$$f(x) > \tfrac{1}{2}f(a) \quad \text{for all } x \in (a - \delta, a + \delta).$$

[Hint: Try taking $\varepsilon = \frac{1}{2}f(a) > 0$ in Definition 1.4.3′.]

SAQ 20 (i) Prove that, if $f: \mathbf{R} \longrightarrow \mathbf{R}$ is continuous at a, then $|f|: \mathbf{R} \longrightarrow \mathbf{R}$ defined by $|f|(x) = |f(x)|$, $x \in \mathbf{R}$, is continuous at a.

(ii) Prove that, if $f: \mathbf{R} \longrightarrow \mathbf{R}$ is continuous at a, and $k \in \mathbf{R}$, then $kf: \mathbf{R} \longrightarrow \mathbf{R}$ defined by $(kf)(x) = kf(x)$, $x \in \mathbf{R}$, is continuous at a.

(iii) Prove that, if $f:\mathbf{R} \longrightarrow \mathbf{R}$ and $g:\mathbf{R} \longrightarrow \mathbf{R}$ are continuous at a, then $f+g:\mathbf{R} \longrightarrow \mathbf{R}$ defined by $(f+g)(x) = f(x) + g(x)$, $x \in \mathbf{R}$, is continuous at a.

*(iv) Prove that, if $f:\mathbf{R} \longrightarrow \mathbf{R}$ and $g:\mathbf{R} \longrightarrow \mathbf{R}$ are continuous at a, then the functions

$$\begin{aligned} \operatorname{Max}\{f, g\}&:x \longmapsto \operatorname{Max}\{f(x), g(x)\}, \quad x \in \mathbf{R}, \\ \operatorname{Min}\{f, g\}&:x \longmapsto \operatorname{Min}\{f(x), g(x)\}, \quad x \in \mathbf{R}, \text{ and} \\ fg&:x \longmapsto f(x)g(x), \quad x \in \mathbf{R}, \end{aligned}$$

are continuous at a.

*(v) Prove that, if $f:\mathbf{R} \longrightarrow \mathbf{R}$ is continuous at a and f never takes the value 0, then the function

$$f : x \longmapsto \frac{1}{f(x)}, \quad x \in \mathbf{R},$$

is continuous at a.

[Hint: You may find it useful to apply the result of SAQ 19 to the function $|f|$.]

SOLUTION TO SAQ 15

Let any $\varepsilon > 0$ be given. We know that we should look at the difference

$$|f(x) - f(1)| = |x^3 - 1| = |x - 1| \cdot |x^2 + x + 1|.$$

For x close to 1, $|x - 1|$ will be small, and $x^2 + x + 1$ approximately equal to 3. So if we restrict our attention to $x \in (0, 2)$,

$$\begin{aligned} |x^2 + x + 1| &\le |x|^2 + |x| + 1 \\ &\le 2^2 + 2 + 1 = 7 \end{aligned}$$

by the triangle inequality; and if we insist also that $x \in (1 - \varepsilon, 1 + \varepsilon)$ we have the bound $|x - 1| < \varepsilon$ on the other factor above. Combining these observations, if we choose $\delta = \operatorname{Min}\{\varepsilon, 1\}$ so that both bounds hold, we see that for all x with $|x - 1| < \delta$,

$$|f(x) - f(1)| < 7\varepsilon.$$

Note (iii) then guarantees that f is continuous at 1. ■

SOLUTION TO SAQ 16

(i) Let $\varepsilon > 0$ be given. Since

$$|f(x) - f(0)| = \begin{cases} |x - 0| = |x|, & \text{if } x \text{ is rational,} \\ |0 - 0| = 0, & \text{if } x \text{ is irrational,} \end{cases}$$

it looks as though $\delta = \varepsilon$ would be a good choice.

Choose $\delta = \varepsilon$. Then, for all x with $|x - 0| < \delta$, it follows that $|f(x) - f(0)| < \varepsilon$ (whether x is rational or irrational); it follows that f is continuous at 0.

(ii) For $a \neq 0$, let (x_n), (y_n) be sequences of rationals, irrationals respectively (different from a), which converge to a. (That such sequences exist is the gist of the rational, irrational density properties.) Then, if $\lim_{x \to a} f(x)$ existed, by S

Lemma 1.3.5 the sequences $(f(x_n))$ and $(f(y_n))$ would have to converge to the same limit. But from the definition of f, $f(x_n) = x_n$ and $f(y_n) = 0$ so that

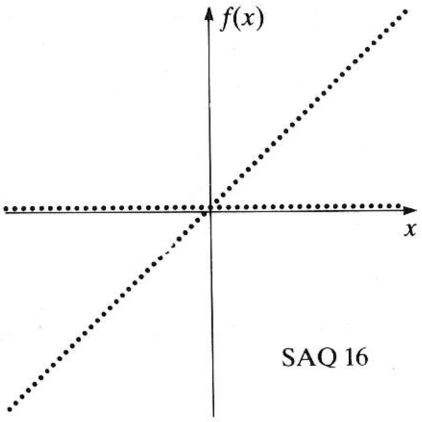
$$\lim_{n \to \infty} f(x_n) = a \quad \text{and} \quad \lim_{n \to \infty} f(y_n) = 0.$$

Since $a \neq 0$ this is impossible, so that f cannot be continuous at a. ■

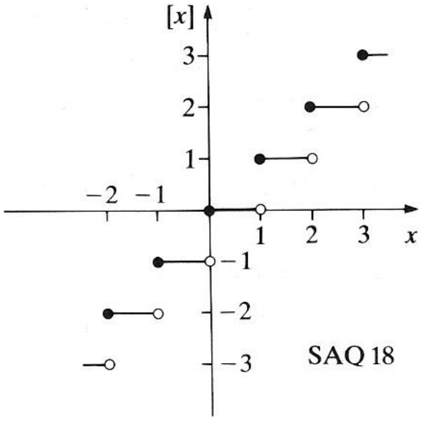
SOLUTION TO SAQ 17

All we have to show is that f and g take the same values at irrational points. So let $a \in \mathbf{R} - \mathbf{Q}$.

Let (x_n) be a sequence of rationals which converges to a. Then by Note (ii), the sequence $(f(x_n))$ converges to $f(a)$ and the sequence $(g(x_n))$ to $g(a)$. But since x_n is rational, and $f = g$ on $\mathbf{Q}$, it follows that $f(x_n) = g(x_n)$ for each n; consequently, by Proposition 1.2.3, $f(a)$ must be equal to $g(a)$, as required. ■



SAQ 16

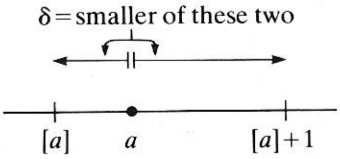


SAQ 18

SOLUTION TO SAQ 18 (i) Let $a \in \mathbf{R} - \mathbf{Z}$. Then there is an integer $[a]$ such that $[a] < a < [a] + 1$; in particular, if we put

$$\delta = \text{Min}\{a - [a], ([a] + 1) - a\},$$

then $\delta > 0$, so that if x is within this distance δ from a, $f(x) = f(a) = [a]$.

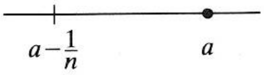


So let $\varepsilon > 0$ be given. Let δ be given by the above formula. Then, for all x with $|x - a| < \delta$, we have that

$$|f(x) - f(a)| = 0 < \varepsilon;$$

so f is continuous at a.

(ii) Let $a \in \mathbf{Z}$, and let $x_n = a - \dfrac{1}{n}$.



If f were continuous at a, the sequence $(f(x_n))$ would have to converge to the limit $f(a)$. However, for each $n \geq 1$, $f(x_n) = \left[a - \dfrac{1}{n}\right] = a - 1$; but $f(a) = a$, so such convergence does not occur in this case. Thus f is not continuous at a. ■

Remark. Remember that to show that f is *not* continuous it is enough to find just *one* sequence (x_n) that does not do what it should.

SOLUTION TO SAQ 19 Take $\varepsilon = \frac{1}{2}f(a) > 0$ in Definition 1.4.3′. Then, since f is continuous at a, there exists some $\delta > 0$ such that

$$|f(x) - f(a)| < \tfrac{1}{2}f(a) \quad \text{whenever } |x - a| < \delta.$$

In other words, if $x \in (a - \delta, a + \delta)$, then

$$-\tfrac{1}{2}f(a) < f(x) - f(a) < \tfrac{1}{2}f(a);$$

rearranging the left inequality gives $f(x) > \frac{1}{2}f(a)$, as required. ■

SOLUTION TO SAQ 20 (i) Let $\varepsilon > 0$ be given. Since f is continuous at a, there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{for all } x \text{ with } |x - a| < \delta.$$

Hence, for all x with $|x - a| < \delta$, we have

$$\begin{aligned} ||f(x)| - |f(a)|| &\leq |f(x) - f(a)| \quad \text{(by Corollary 1.1.10)} \\ &< \varepsilon \quad \text{(as above);} \end{aligned}$$

this is exactly what we need for $|f|$ to be continuous at a. (So the 'cunning form' of the triangle inequality was the only tool we needed.)

(ii) Let $\varepsilon > 0$ be given. Since f is continuous at a, there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{for all } x \text{ with } |x - a| < \delta;$$

consequently,

$$|kf(x) - kf(a)| < |k|\varepsilon \quad \text{for all } x \text{ with } |x - a| < \delta.$$

If $k \neq 0$, it follows, by Note (iii), that f is continuous at a; whereas if $k = 0$,

$$|kf(x) - kf(a)| = 0 < \varepsilon \quad \text{for all } x,$$

so that in this case too, f is continuous at a.

(iii) Let $\varepsilon > 0$ be given. Since f is continuous at a, there exists $\delta_1 > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{for all } x \text{ with } |x - a| < \delta_1;$$

and since g is continuous at a, there exists $\delta_2 > 0$ such that

$$|g(x) - g(a)| < \varepsilon \quad \text{for all } x \text{ with } |x - a| < \delta_2.$$

So if we choose $\delta = \text{Min}\,\{\delta_1, \delta_2\}$, both inequalities will hold; in other words, for all x with $|x - a| < \delta$,

$$|f(x) - f(a)| < \varepsilon \quad \text{and} \quad |g(x) - g(a)| < \varepsilon.$$

We know that we have to look at $|(f + g)(x) - (f + g)(a)|$, and notice that we can regroup these terms as $|\{f(x) - f(a)\} + \{g(x) - g(a)\}|$—which suggests at once that the triangle inequality is going to relate this difference to the expressions we have just been looking at. So, for all x with $|x - a| < \delta$, we see that

$$\begin{aligned} |(f + g)(x) - (f + g)(a)| &= |\{f(x) - f(a)\} + \{g(x) - g(a)\}| \\ &\leq |f(x) - f(a)| + |g(x) - g(a)| \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Note (iii) then shows that $f + g$ is continuous at a.

(iv) By SAQ 3,

$$\text{Max}\,\{f(x), g(x)\} = \tfrac{1}{2}\{f(x) + g(x) + |f(x) - g(x)|\}.$$

Therefore

$$\text{Max}\{f, g\} = \tfrac{1}{2}\{f + g + |f - g|\}.$$

By part (ii) with $k = -1$, $-g$ is continuous at a; and so, by part (iii), $f - g$ is continuous at a. Hence so is $|f - g|$ by part (i). Using part (iii) twice in turn, we see that $g + |f - g|$ and then $f + g + |f - g|$ is continuous at a. Finally, if we use part (ii) with $k = \frac{1}{2}$, we get that $\text{Max}\,\{f, g\}$ is continuous at a.

A similar argument works for

$$\text{Min}\,\{f(x), g(x)\} = \tfrac{1}{2}\{f(x) + g(x) - |f(x) - g(x)|\}.$$

The argument for fg is rather different, and depends on a 'grouping of terms' trick which is slightly more cunning than the grouping trick in (iii).

(v) By SAQ 19, there exists some $\delta > 0$ such that

$$|f(x)| > \tfrac{1}{2}|f(a)| \quad \text{for all } x \text{ with } |x - a| < \delta.$$

Let $\varepsilon > 0$ be given (as usual at this stage!).

Then, since we were told that f is continuous at a, there exists $\delta_1 > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{for all } x \text{ with } |x - a| < \delta_1.$$

Again as usual, let us arrange for both our inequalities for $f(x)$ to hold by choosing $\delta' = \text{Min}\,\{\delta, \delta_1\}$ and looking only at the interval $(a - \delta', a + \delta')$.

Then, for all x with $|x - a| < \delta'$,

$$\left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| = \frac{|f(x) - f(a)|}{|\ (x)|\,|f(a)|} \qquad \text{(an obviously useful rearrangement)}$$

$$< \frac{|f(x)-f(a)|}{\frac{1}{2}|f(a)|\,|f(a)|} \qquad \text{(by our first inequality)}$$

$$< \frac{\varepsilon}{\frac{1}{2}|f(a)|^2} \qquad \text{(by our second inequality).}$$

Hence, by Note (iii), $\dfrac{1}{f}$ is continuous at a. ■

2.5 Objectives of Unit 2

After studying the unit, you should

(1) understand what the least upper bound and greatest lower bound of a set are, and be able to determine their values;

(2) know and be able to use the rational and irrational density properties;

(3) know and be able to use both versions of the triangle inequality;

(4) understand the Archimedean Axiom;

(5) know the definition of convergence of a sequence, and be able to prove that a given sequence converges or that it does not converge;

(6) know the basic properties of convergent sequences;

(7) know the properties of monotonic and Cauchy sequences;

(8) fully understand the $K\varepsilon$ lemma (**S** Lemma 1.2.4, etc.) in its various forms;

(9) know the definition of $\lim\limits_{x \to a} f(x)$, and be able to prove that a given function tends to a limit or does not tend to a limit;

(10) know the relationship between limits and sequences (**S** Lemma 1.3.5), and be able to use it;

(11) know the definition of continuity, both in terms of limit and in terms of ε and δ, and be able to prove that a given function is or is not continuous at a point.

2.6 Revision Questions

Remark on terminology in SAQ 21. A set X in $\mathbf{R}$ is said to be *bounded* if it is both bounded above and bounded below. It follows from $(*)$ in Note (xii) of Section 2.1 that X is bounded if and only if some $K \geq 0$ exists such that $|x| \leq K$ for all $x \in X$.

SAQ 21 Let X be a non-empty bounded subset of $\mathbf{R}$, and $Y = \{|x| : x \in X\}$.

(i) Show that Y is a bounded subset of $\mathbf{R}$.

(ii) Prove that

$$\sup Y = \text{Max}\,\{|\sup X|, |\inf X|\}.$$

Without providing detailed proofs, verify that this identity holds for the following examples of Xs:

$$[2,3], [-1,2], [-2,1], [-3,-2].$$

(iii) (It is possible in the same way as in (ii) to prove that

$$0 \leq \inf Y \leq \text{Min}\,\{|\sup X|, |\inf X|\}.)$$

Write down a specific subset X of $\mathbf{R}$ for which

$$0 < \inf Y < \text{Min}\,\{|\sup X|, |\inf X|\}.$$

SAQ 22 Exercise 1.5, number 10, **S** page 16.

SAQ 23 Exercise 1.5, number 12, **S** page 17.

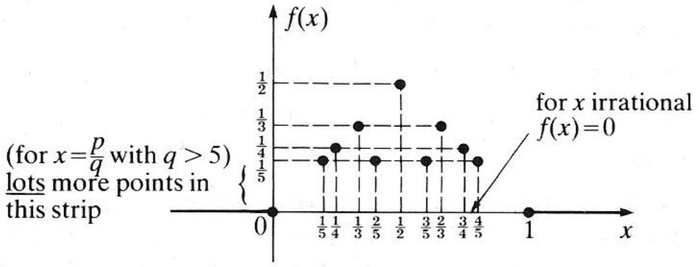
SAQ 24 Prove carefully that the function

$$f: x \longmapsto \begin{cases} 0, & \text{if } x = 0, \\ \dfrac{1}{x}, & \text{if } x = 0, \end{cases}$$

is (i) continuous at 2, (ii) not continuous at 0.

***SAQ 25** Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be defined as follows:

$f(x) = 0$ if x is irrational or $x \leq 0$ or $x \geq 1$, and $f(x) = \dfrac{1}{q}$ if x is a rational number in $(0, 1)$, where $x = p/q$ with p and q positive integers with no common factor greater than 1.



Prove that $\lim_{x \to a} f(x) = 0$ for every $a \in \mathbf{R}$. For which $a \in \mathbf{R}$ is f is continuous at a?

SOLUTION TO SAQ 21

(i) Since X is bounded, there exists some real number K such that

$$|x| \leq K \quad \text{for all } x \in X;$$

and so such that

$$y \leq K \quad \text{for all } y \in Y.$$

Since $y \geq 0$ (as $|x| \geq 0$), it follows that $0 \leq y \leq K$ and so Y is bounded.

(ii) Let $a = \inf X$ and $b = \sup X$.

Suppose first that $a \geq 0$. Then $x \geq 0$ for all $x \in X$, so that $Y = X$. Then

$$\sup Y = \sup X;$$

and since $0 \leq \inf X \leq \sup X$ and $\inf X = |\inf X|$ and $\sup X = |\sup X|$, it follows that

$$\sup Y = \text{Max}\,\{|\inf X|, |\sup X|\}.$$

Suppose next that $b \leq 0$. Then

$$a \leq x \leq b \leq 0 \quad \text{for all } x \text{ in } X.$$

By Example 2 in the special case $k = -1$, it follows that

$$\begin{aligned} \sup Y &= -\inf X \\ &= |\inf X| \\ &= \text{Max}\,\{|\inf X|, |\sup X|\} \qquad (\text{since} |a| \geq |b|). \end{aligned}$$

Finally, we have the case where $a < 0$ and $b > 0$. If $x \in X$ and $x \geq 0$, then $|x| = x \leq b$; and if $x \in X$ and $x < 0$, then $|x| = -x \leq -a$.

Hence

$$\begin{aligned} |x| &\leq \text{Max}\,\{b, -a\} \\ &= \text{Max}\,\{|b|, |a|\}, \end{aligned}$$

so that $\text{Max}\,\{|b|, |a|\}$ is *an* upper bound for Y.

But if u is an upper bound for Y, then $|x| \leq u$ for all $x \in X$; so that $-u \leq x \leq u$ for all $x \in X$. Hence $\sup X \leq u$ and $\inf X \geq -u$; it follows from the former inequality that

$$(|b| =)\, b \leq u$$

and from the latter that

$$(|a| =) - a \leq u.$$

Thus $u \geq \text{Max}\,\{|a|, |b|\}$; so that $\text{Max}\,\{|a|, |b|\}$ is the least upper bound of Y, as required.

X	Y	$\lvert\inf X\rvert$	$\lvert\sup X\rvert$	$\text{Max}\,\{\lvert\inf X\rvert, \lvert\sup X\rvert\}$	$\sup Y$
$[2,3]$	$[2,3]$	2	3	3	3
$[-1,2]$	$[0,2]$	1	2	2	2
$[-2,1]$	$[0,2]$	2	1	2	2
$[-3,-2]$	$[2,3]$	3	2	3	3

(iii) Let $X = [-3, -1] \cup [2, 4]$. (Notice that X omits everything near 0—that is the trick in thinking of such a weird example!) The corresponding set Y is the closed interval $[1, 4]$, so that $\inf Y = 1 > 0$; and

$$\begin{aligned}\text{Min}\,\{|\sup X|, |\inf X|\} &= \text{Min}\,\{4, 3\} \\ &= 3 > 1 = \inf Y. \quad ■\end{aligned}$$

SOLUTION TO SAQ 22

We are to show that $\lim_{n\to\infty} n^{1/n} = 1$.

If we put $a_n = n^{1/n} - 1$, then it is enough to prove that the sequence (a_n) converges to 0.

Notice that when $n > 1$ we have $n^{1/n} > 1$; and so, for $n > 1$, $a_n > 0$. Then, if we use the Binomial Theorem,

$$\begin{aligned} n &= (1 + a_n)^n \\ &= 1 + na_n + \tfrac{1}{2}n(n-1)a_n^2 + \cdots + a_n^n \\ &\geq 1 + \tfrac{1}{2}n(n-1)a_n^2 \qquad (\text{for } n \geq 2), \end{aligned}$$

or

$$n - 1 > \tfrac{1}{2}n(n-1)a_n^2 \qquad (\text{for } n \geq 2).$$

Dividing this inequality by $(n - 1)$ and rearranging, we deduce that $a_n^2 < 2/n$, so that $0 < a_n < \sqrt{2}/n^{1/2}$.

But SAQ 5 showed that $\lim_{n\to\infty} 1/n^{1/2} = 0$.

So now let $\varepsilon > 0$ be given. Then there exists some integer $N \geq 2$ such that $|1/n^{1/2}| < \varepsilon$ for all $n \geq N$. It follows that

$$|a_n| < \sqrt{2}\varepsilon \quad \text{for all } n \geq N,$$

so that (by Lemma 1.2.4 with $K = \sqrt{2}$) (a_n) converges to 0. ■

SOLUTION TO SAQ 23

Let $\varepsilon > 0$ be given. Since (s_{2n}) converges to l, there exists an N_1 such that

$$|s_{2n} - l| < \varepsilon \quad \text{for all } n \geq N_1;$$

and since (s_{2n+1}) converges to l, there exists an N_2 such that

$$|s_{2n+1} - l| < \varepsilon \quad \text{for all } n \geq N_2.$$

(For sufficiently large n, s_n will satisfy one or other of these inequalities; let us make this precise!)

Let $N = \text{Max}\,\{2N_1, 2N_2 + 1\}$. If n is an integer such that $n \geq N$, then either n is even or n is odd. If n is even, then $n = 2m$ where $m \geq N_1$ (since $n \geq 2N_1$), so that

$$|s_n - l| = |s_{2m} - l| < \varepsilon;$$

and if n is odd, then $n = 2m + 1$ where $m \geq N_2$ (since $n \geq 2N_2 + 1$), so that

$$|s_n - l| = |s_{2m+1} - l| < \varepsilon.$$

This argument shows that

$$|s_n - l| < \varepsilon \quad \text{for all } n \geq N. \quad ■$$

SOLUTION TO SAQ 24

(i) Let $\varepsilon > 0$ be given. Then

$$|f(x) - f(2)| = \left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{2|x|}.$$

Now if we restrict our attention to those x that satisfy the inequality $|x - 2| < 1$, then $x \in (1, 3)$ so that $|x| > 1$ and $\frac{1}{|x|} < 1$.

So now choose $\delta = \text{Min}\{1, \varepsilon\}$.

Then for all x with $|x - 2| < \delta$,

$$|f(x) - f(2)| < \frac{\varepsilon}{2 \times 1} < \varepsilon.$$

Therefore f is continuous at 2.

(ii) (We use our contradiction argument technique here.) Suppose that f were continuous at 0.

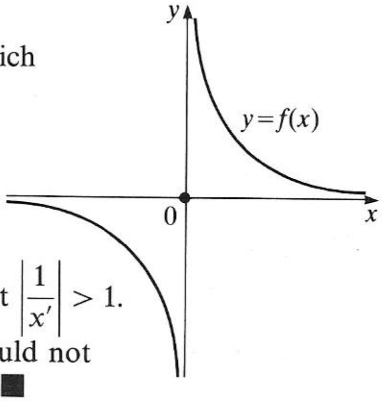
Then, taking $\varepsilon = 1$ in Definition 1.4.3′, there exists $\delta > 0$ such that

$$|f(x) - f(0)| < 1 \quad \text{for all } x \text{ with } |x - 0| < \delta;$$

in other words,

$$\left|\frac{1}{x}\right| < 1 \quad \text{for all } x \text{ with } |x| < \delta. \qquad (*)$$

We now want to find an x with $|x| < \delta$ for which the inequality $\left|\frac{1}{x}\right| < 1$ does not hold.



So, choose $x' = \frac{1}{2}\text{Min}\{\delta, 1\}$; for x', $|x'| < \delta$ but $\left|\frac{1}{x'}\right| > 1$.

This contradicts the assertion $(*)$; and so f could not have been continuous at 0 in the first place. ■

SOLUTION TO SAQ 25

It is clear that $\lim_{x \to a} f(x) = 0$ if $a < 0$ or $a > 1$.

Now suppose that $0 \le a \le 1$; and let $\varepsilon > 0$ be given. Let N be the least positive integer such that $N > \frac{1}{\varepsilon}$. Then there are only finitely many rational numbers x such that $x = p/q$ where p and q are positive integers with no common factor greater than 1 and with $p < q < N$; so let x_0 be the rational number of this form which is nearest to a (i.e. such that $|x_0 - a|$ is least), and take $\delta = |x_0 - a|$.

Then if $0 < |x - a| < \delta$ and x is not a rational number in $(0, 1)$, $f(x) = 0$. On the other hand, if $0 < |x - a| < \delta$ and $x = p/q$ where p, q are positive integers with no common factor greater than 1, then $q \ge N$ so that

$$f(x) = \frac{1}{q} \le \frac{1}{N} < \varepsilon.$$

Combining these two cases we conclude that

$$|f(x)| < \varepsilon \quad \text{for } \textit{all } x \text{ with } 0 < |x - a| < \delta.$$

Therefore $\lim_{x \to a} f(x) = 0$.

By Definition 1.4.3, f is continuous at a whenever $\lim_{x \to a} f(x)$ equals $f(a)$. Since $\lim_{x \to a} f(x) = 0$ for every a, the function f is continuous at a for those a such that $f(a) = 0$. Thus f is continuous at a if a is irrational, or if $a \le 0$, or if $a \ge 1$. ■

UNIT 3

METRIC SPACES AND CONTINUITY

3.0 Introduction

In this unit, which is based on Chapter 2 of **S**, we study *metric spaces*. A metric space consists of a set of points together with a function, called a *metric*, which assigns to each pair of points a non-negative real number which we call the *distance* between them. The definition of a metric is arrived at by abstraction from the properties of the familiar distance in the Euclidean plane or in three-space, and Euclidean spaces are important examples of metric spaces. Abstract metric spaces may, however, have some unexpected properties and be less readily interpreted intuitively. Nevertheless, we expect that you will find the concept of a metric space simple and geometrically clear.

A rich area for applications of metric space theory is in the branch of advanced analysis called *functional analysis*. In Section 3.4 we take a brief look at the types of space which arise in these applications. We shall return to this topic in *Unit 8*.

One reason for introducing metric spaces is to provide a good setting for studying continuity of functions. Intuitively, a function f is continuous at a if the images under f of all points sufficiently close to a in the domain are close to $f(a)$ in the codomain. In making this idea precise, we obtain a definition of continuity of functions between metric spaces which is a natural generalization of the ε-δ definition of continuity of real functions on $\mathbf{R}$ considered in *Unit 2*, Section 2.4.

In Section 3.5, certain subsets of a metric space called *open sets* are identified. The properties of open sets form the basis for a further process of abstraction which will give us the objects known as *topological spaces* to be studied in *Unit 4* and subsequent units.

3.1 Metrics and Metric Spaces

In this section, metric spaces are defined and we look at some examples.

READ S from the beginning of Chapter 2, page 19 to the end of Example 2.2.4, page 24, line −11.

NOTES

(i) *Page 21, Definition 2.1.2*
As a result of the following observation, some trivial cases can be omitted in checking that a function $d\colon A \times A \to \mathbf{R}$ which satisfies (M1) is a metric. To establish (M3), it is enough to show that if $x, y, z \in A$ are all different, then $d(x,z) \leq d(x,y) + d(y,z)$. The triangle inequality follows from (M1) if the points x, y and z are not all distinct. To see this suppose first that $x = y$. Then

$$d(x,z) = d(y,z) = d(x,y) + d(y,z)$$

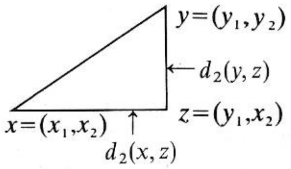
since $d(x,y) = 0$. If $y = z$ the argument is similar. Lastly, if $x = z$ then

$$d(x,z) = 0 \leq d(x,y) + d(y,z).$$

(ii) *Page 23, Example 2.2.2*
S has motivated the definition of a metric by considering ordinary Euclidean distance in the plane and in three-space. We *stress* that a metric space consists of an arbitrary non-empty set A together with a function $d\colon A \times A \to \mathbf{R}$ satisfying the conditions (M1), (M2) and (M3). The elements

of A, that is the points of the metric space $\{A, d\}$, need not be interpretable as physical 'points'. In Section 3.4 we shall consider important examples of metric spaces whose points are functions. Also, in Examples 1 and 2 below, we consider metrics on $\mathbf{R}$ and $\mathbf{R}^2$, respectively, that are different from the Euclidean metrics, and we find that our idea of 'nearness' of points has to be revised. This is the case also for the discrete metric (**S** Example 2.2.2) on the set $\mathbf{R}$. For if $a \in \mathbf{R}$ and $0 < r \leq 1$ then there are no points of $\mathbf{R}$ different from a with 'distance' from a less than r. The point to note is that it is functions satisfying (M1), (M2) and (M3) that are under consideration.

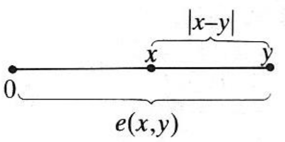
(iii) *Page 24, Example 2.2.3*
The metrics d_1, d_2 and d_∞ on $\mathbf{R}^2$ can be interpreted geometrically. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points of $\mathbf{R}^2$. Then $d_2(x, y)$ is the Euclidean distance between x and y.



To see what d_1 and d_∞ mean geometrically, consider a third point $z = (y_1, x_2)$, so that x,y and z are the vertices of a right-angled triangle. Then $|x_1 - y_1| = d_2(x, z)$ and $|x_2 - y_2| = d_2(z, y)$. Thus $d_1(x, y)$ is the sum of the Euclidean distances $d_2(x, z)$ and $d_2(z, y)$. For this reason, d_1 is sometimes called the 'taxi-cab metric' on $\mathbf{R}^2$ as $d_1(x, y)$ is the distance travelled between the points x and y by a taxi-cab in a North-American city with its rectangular grid of roads. Also, $d_\infty(x, y)$ is the larger of the Euclidean distances $d_2(x, z)$ and $d_2(z, y)$. We can now see that if we use d_1 or d_∞ as a measure of distance in $\mathbf{R}^2$ then points are close if and only if they are close in the Euclidean sense, i.e. with the metric d_2 as a measure of distance. This will be seen later to be the essence of the remark in **S** (page 24, line 15) about the equivalence of d_1, d_2 and d_∞. ▲

EXAMPLE 1 The function $e: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by

$$e(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \text{Max}\{|x|, |y|\}, & \text{if } x \neq y, \end{cases}$$



is a metric on $\mathbf{R}$.

Clearly, $e(x, y) \geq 0$ for all x, y in $\mathbf{R}$. If $x = y$ then $e(x, y) = 0$ by definition. Conversely, if $x \neq y$ then x and y are not both zero so that $\text{Max}\{|x|, |y|\} > 0$. Thus (M1) is satisfied. It is immediate from the definition that $e(x, y) = e(y, x)$ for all x, y in $\mathbf{R}$ so (M2) is satisfied. Finally, we must verify that (M3) holds, i.e. that

$$e(x, z) \leq e(x, y) + e(y, z)$$

for all x, y, z in $\mathbf{R}$. But

$$\begin{aligned} e(x, z) = \text{Max}\{|x|, |z|\} &\leq |x| + |z| \\ &\leq \text{Max}\{|x|, |y|\} + \text{Max}\{|y|, |z|\} \\ &= e(x, y) + e(y, z). \end{aligned}$$

Thus (M3) holds and this completes the proof that e is a metric on $\mathbf{R}$.

Notice that if $a \neq 0$ and $x \neq a$ then $e(a, x) \geq |a| > 0$. Thus if r is a real number such that $0 < r \leq |a|$ then

$$\{x: e(a, x) < r\} = \{a\};$$

i.e. the only point whose e-distance from $a \neq 0$ is less than r is a itself. Also $e(0, x) = |x|$ for all x in $\mathbf{R}$, so that

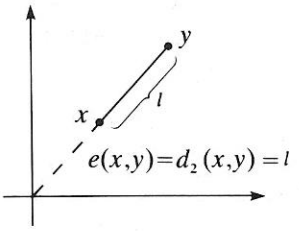
$$\{x: e(0, x) < r\} = \{x: |x| < r\} = (-r, r)$$

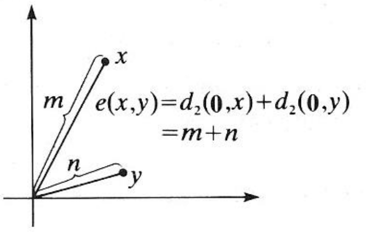
for all real numbers $r > 0$.

EXAMPLE 2 Let $e:\mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ be given by

$$e(x,y) = \begin{cases} 0, & \text{if } x = y, \\ d_2(x,y), & \text{if } x \neq y \text{ and } x \text{ and } y \text{ are collinear with } \mathbf{0}, \\ d_2(\mathbf{0},x) + d_2(\mathbf{0},y), & \text{otherwise,} \end{cases}$$

where d_2 is the Euclidean metric on $\mathbf{R}^2$ and $\mathbf{0} = (0,0)$.





Then e is a metric on $\mathbf{R}^2$.

To start with, observe that $e(x,y) \geq 0$ for all x, y in $\mathbf{R}^2$; also $e(x,x) = 0$ for all x in $\mathbf{R}^2$ and $e(x,y) > 0$ if $x, y \in \mathbf{R}^2$ and $x \neq y$. Thus condition (M1) is satisfied. Clearly, condition (M2) is satisfied. Now let x, y and z be points of $\mathbf{R}^2$ which are all different. To check that the condition

$$e(x,z) \leq e(x,y) + e(y,z)$$

is satisfied we consider four cases: (i) x, y and z are all collinear with $\mathbf{0}$; (ii) y and z are collinear with $\mathbf{0}$ but x, y and z are not collinear; (iii) x and z are collinear with $\mathbf{0}$ but x, y and z are not collinear; (iv) no two of x, y and z are collinear with $\mathbf{0}$.

In case (i), using the triangle inequality for d_2 we have

$$e(x,z) = d_2(x,z) \leq d_2(x,y) + d_2(y,z) = e(x,y) + e(y,z).$$

In case (ii),

$$e(x,z) = d_2(\mathbf{0},x) + d_2(\mathbf{0},z).$$

By (M3), $d_2(\mathbf{0},z) \leq d_2(\mathbf{0},y) + d_2(y,z)$. Also $e(x,y) = d_2(\mathbf{0},x) + d_2(\mathbf{0},y)$ and $e(y,z) = d_2(y,z)$. Thus

$$e(x,z) \leq d_2(\mathbf{0},x) + d_2(\mathbf{0},y) + d_2(y,z) = e(x,y) + e(y,z).$$

We leave you to check cases (iii) and (iv) as an exercise, thus establishing that e is a metric on $\mathbf{R}^2$.

Notice that points which are near in the Euclidean sense may be far apart with respect to the metric e. For example, if $x = (0,1)$ and $y = (10^{-3},1)$, then $d_2(x,y) = 10^{-3}$ whilst

$$e(x,y) = d_2(\mathbf{0},x) + d_2(\mathbf{0},y) = 1 + (1 + 10^{-6})^{\frac{1}{2}} > 2.$$

We shall return to the study of this example later in the unit.

SAQ 1 Which of the following functions d: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are metrics on $\mathbf{R}$?

(i) $d(x,y) = 3|x - y|$, (ii) $d(x,y) = x^2 + y^2$,

(iii) $d(x,y) = (x - y)^2$, (iv) $d(x,y) = x - y$,

where $(x,y) \in \mathbf{R} \times \mathbf{R}$.

SAQ 2 State which of the conditions (M1), (M2) and (M3) are satisfied by the following functions d: $\mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$.

(i) $d(x,y) = \text{Max } \{2|x_1 - y_1|, 3|x_2 - y_2|\}$,

(ii) $d(x,y) = |x_1 - y_1|\,|x_2 - y_2|$,

where $x = (x_1, x_2)$, $y = (y_1, y_2)$.

SAQ 3 Let $\{A, d\}$ be a metric space with more than one point.

(i) For what values of the real number k is it true that d_k, where

$$d_k\colon (x,y) \mapsto k\,d(x,y), \qquad (x,y) \in A \times A,$$

is a metric on A?

(ii) Prove that $\bar{d}$, where

$$\bar{d}\colon (x,y) \mapsto \text{Min}\,\{1, d(x,y)\}, \qquad (x,y) \in A \times A,$$

is a metric on A.

SAQ 4 Let x, y and z be points of a metric space $\{A, d\}$. Prove that

$$|d(x,z) - d(y,z)| \le d(x,y).$$

Interpret this result in the case that $A = \mathbf{R}$, d is the Euclidean metric and $z = 0$.

SOLUTION TO SAQ 1

(i) Since $3|x - y| \ge 0$ for all x, y in $\mathbf{R}$ and $3|x - y| = 0$ if and only if $x = y$, condition (M1) is satisfied. Clearly, (M2) is satisfied. If x, y, $z \in \mathbf{R}$ then

$$3(x - z) = 3\big((x - y) + (y - z)\big)$$

so that

$$3|x - z| \le 3|x - y| + 3|y - z|$$

and hence (M3) is satisfied. Thus d is a metric on $\mathbf{R}$.

(ii) If $x \ne 0$ then $d(x,x) = 2x^2 > 0$ so that (M1) is not satisfied and so d is not a metric.

(iii) This is not a metric since (M3) fails to hold. If $x = 1$, $y = 0$ and $z = -1$ then $d(x,z) = 4$, $d(x,y) = 1$ and $d(y,z) = 1$, so that it is not true that

$$d(x,z) \le d(x,y) + d(y,z).$$

Notice that one numerical counter-example suffices to show that (M3) does not hold.

(iv) If $x < y$ then $d(x,y) < 0$ so that (M1) is not satisfied. ■

SOLUTION TO SAQ 2

(i) By an argument similar to the verification that d_∞ is a metric on $\mathbf{R}^2$ (S Example 2.2.3), we see that d satisfies (M1), (M2) and (M3).

(ii) If $x = (0,0)$ and $y = (0,1)$ then $x \ne y$ but $d(x,y) = 0$. Thus d does not satisfy condition (M1). Clearly, d satisfies (M2). If $x = (1,0)$, $y = (1,1)$ and $z = (0,1)$ then $d(x,z) = 1$ but $d(x,y) = 0$ and $d(y,z) = 0$ so that (M3) is not satisfied. ■

SOLUTION TO SAQ 3

(i) If $k < 0$ then $d_k(x,y) < 0$ if $x, y \in A$ and $x \ne y$; and if $k = 0$ then $d_k(x,y) = 0$ for all x, y in A. Thus d_k does not satisfy (M1) if $k \le 0$. (Notice that we have used the fact that A contains more than one point, and so such unequal x and y.)

Now we show that if $k > 0$ then d_k is a metric on A. Since $d(x,y) \ge 0$ for all x, y in A, it follows that $d_k(x,y) \ge 0$ for all x, y in A. Also $d_k(x,y) = k\,d(x,y) = 0$ if and only if $d(x,y) = 0$, which is the case if and only if $x = y$. Thus (M1) holds. If $x, y \in A$ then

$$d_k(x,y) = kd(x,y) = kd(y,x) = d_k(y,x)$$

so that (M2) holds. If x, y, $z \in A$ then

$$d(x,z) \le d(x,y) + d(y,z).$$

Since $k > 0$ it follows that

$$kd(x,z) \le kd(x,y) + kd(y,z)$$

and so d_k satisfies (M3). Therefore d_k is a metric on A if and only if $k > 0$.

(ii) It is easy to see that $\bar{d}$ satisfies conditions (M1) and (M2). Let $x, y, z \in A$. Since $\bar{d}(x, z) \leq 1$, the required inequality

$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$$

is satisfied if either $\bar{d}(x, y) = 1$ or $\bar{d}(y, z) = 1$. The only other possibility is that $\bar{d}(x, y) < 1$ and $\bar{d}(y, z) < 1$. In this case both $\bar{d}(x, y) = d(x, y)$ and $\bar{d}(y, z) = d(y, z)$ so that

$$\bar{d}(x, z) \leq d(x, z) \leq d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z).$$

Hence (M3) is satisfied. Thus $\bar{d}$ is a metric on A. ■

SOLUTION TO SAQ 4 Since $d(x, z) \leq d(x, y) + d(y, z)$ it follows that

$$d(x, z) - d(y, z) \leq d(x, y).$$

Interchanging x and y in this inequality, we see that

$$d(y, z) - d(x, z) \leq d(y, x) = d(x, y)$$

so that

$$d(x, z) - d(y, z) \geq -d(x, y).$$

Thus

$$-d(x, y) \leq d(x, z) - d(y, z) \leq d(x, y),$$

from which it follows that

$$|d(x, z) - d(y, z)| \leq d(x, y).$$

If $A = \mathbf{R}$, d is the Euclidean metric and $z = 0$ then $d(x, z) = |x|$, $d(y, z) = |y|$ and $d(x, y) = |x - y|$. Thus we obtain the inequality

$$\big||x| - |y|\big| \leq |x - y|. \quad ■$$

3.2 Subspaces and Products

Given a metric space $\{A, d\}$ and a subset H of A, there is a natural metric on H. Also, given metric spaces $\{A, d\}$ and $\{A', d'\}$ there are several ways of combining d and d' to obtain metrics on the Cartesian product $A \times A'$.

READ S Example 2.2.5 (page 24, line −7 to page 25, line 6) and Example 2.2.7 (page 26, lines 1 to 11).

NOTES (i) *Page 24, Example 2.2.5*

The restriction of a function to a subset of its domain was considered in Section 1.3 of *Unit 1*. If $\{A, d\}$ is a metric space then $d\colon A \times A \to \mathbf{R}$ is a function satisfying conditions (M1), (M2) and (M3). If $H \subset A$ then $H \times H \subset A \times A$ and we can define $d_H\colon H \times H \to \mathbf{R}$ by putting $d_H = d|H \times H$.

(ii) *Page 25, line 2*

S adopts the convention that if a subset H of $\mathbf{R}^n$ is referred to as a metric space then it is to be assumed that the metric on H is induced by the Euclidean metric on $\mathbf{R}^n$. It is convenient to refer to this metric as the *Euclidean metric* on H.

(iii) *Page 26, Example 2.2.7*

S uses the same notation d_1, d_2 and d_∞ for metrics on a Cartesian product as used previously, in Example 2.2.3, for metrics on $\mathbf{R}^2$. Since $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$, Example 2.2.3 is the special case of Example 2.2.7 in which M and M' are taken to be $\mathbf{R}$ with the Euclidean metric. ▲

EXAMPLE 3 Let H be a line through $\mathbf{0} = (0,0)$ in $\mathbf{R}^2$. The metric induced on H by the metric e on $\mathbf{R}^2$, introduced in Example 2, is the Euclidean metric on H. For if $x, y \in H$ then $e(x,y) = d_2(x,y)$, by definition of H and e, where d_2 is the Euclidean metric on $\mathbf{R}^2$. Re-examination of the last part of Example 2 will show, however, that the metric induced by e on the 'horizontal' line

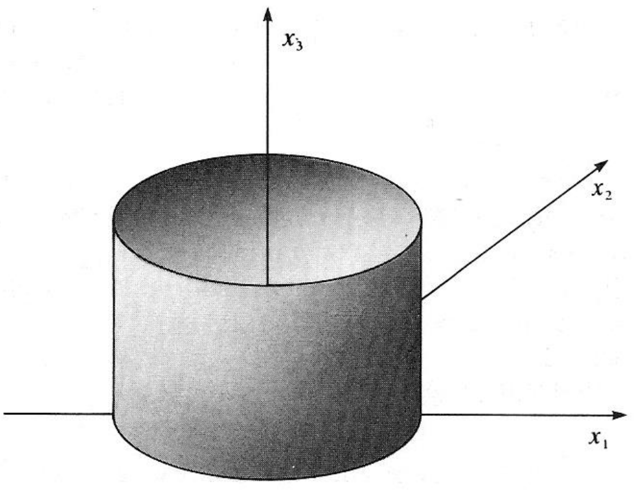
$$K = \{(x_1, x_2) \in \mathbf{R}^2 \colon x_2 = 1\}$$

is very (!) different from the Euclidean metric on K.

EXAMPLE 4 Let

$$S^1 = \{(x_1, x_2) \in \mathbf{R}^2 \colon x_1^2 + x_2^2 = 1\}$$

be the unit circle in $\mathbf{R}^2$. Each of the metrics d_1, d_2 and d_∞ on $\mathbf{R}^2$ induces a metric on S^1. In particular, the Euclidean metric on S^1 is that induced by d_2.



Now let us look at the cylinder segment

$$C = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \colon x_1^2 + x_2^2 = 1, 0 \leq x_3 \leq 1\}.$$

There are several metrics on C which we might consider. Firstly, $C \subset \mathbf{R}^3$ and so has a Euclidean metric. But $C = S^1 \times [0,1]$ so that if $[0,1]$ has its Euclidean metric and S^1 has one of the metrics considered above, then metrics can be defined on C in accordance with Example 2.2.7. You should check that if S^1 and $[0,1]$ have the Euclidean metrics then the d_2-metric of Example 2.2.7 on $C = S^1 \times [0,1]$ is the Euclidean metric.

SAQ 5 Let M be the metric space $\{\mathbf{R}, e\}$ considered in Example 1 and let M' be $\mathbf{R}$ with the Euclidean metric. Let d_1, d_2 and d_∞ be the product metrics defined on $\mathbf{R} \times \mathbf{R}$ as in **S** Example 2.2.7, and let $x = (1,2)$ and $y = (3,7)$ in $\mathbf{R} \times \mathbf{R}$. Find $d_1(x,y)$, $d_2(x,y)$ and $d_\infty(x,y)$.

SAQ 6 Let $\{A, d\}$ and $\{A', d'\}$ be discrete metric spaces (**S** Example 2.2.2). Prove that the product metric d_∞ on $A \times A'$ is the discrete metric.

SAQ 7 Find the product metric 'd_1' of **S** Example 2.2.7 on $\mathbf{R} \times \mathbf{R}$ when the metric on each of the factors is the metric e on $\mathbf{R}$ introduced in Example 1.

SOLUTION TO SAQ 5 Since $e(1,3) = 3$ it follows that $d_1(x,y) = 3 + 5 = 8$, $d_2(x,y) = \sqrt{(3^2 + 5^2)}$ $= \sqrt{(34)}$ and $d_\infty(x,y) = \text{Max}\{3,5\} = 5$. ■

SOLUTION TO SAQ 6 Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of $A \times A'$. If $x = y$ then $d_\infty(x,y) = 0$ since d_∞ is a metric. If $x \neq y$ then either $x_1 \neq y_1$ or $x_2 \neq y_2$. Thus either $d(x_1, y_1) = 1$ or $d'(x_2, y_2) = 1$. Since d and d' take only the values 0 and 1 we see that

$$d_\infty(x,y) = \text{Max}\{d(x_1, y_1), d'(x_2, y_2)\} = 1.$$

Thus d_∞ is the discrete metric on $A \times A'$. ■

SOLUTION TO SAQ 7 Let us denote the required metric on $\mathbf{R} \times \mathbf{R}$ by e_1 (we should not use d_1 since this has an established meaning for $\mathbf{R}^2$). If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ then

$$e_1(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \operatorname{Max}\{|x_2|, |y_2|\}, & \text{if } x_1 = y_1, x_2 \neq y_2, \\ \operatorname{Max}\{|x_1|, |y_1|\}, & \text{if } x_1 \neq y_1, x_2 = y_2, \\ \operatorname{Max}\{|x_1|, |y_1|\} + \operatorname{Max}\{|x_2|, |y_2|\}, & \text{if } x_1 \neq y_1, x_2 \neq y_2. \end{cases} \quad ■$$

3.3 Continuity

READ S all of Section 2.1 (again).

NOTES (i) *Page 19, line 1*
We have asked you to re-read Section 2.1 because of the importance of the ideas introduced there. In this section we are going to concentrate on continuity of functions whose domain and codomain are metric spaces.

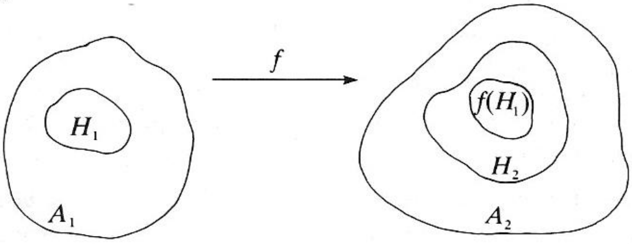
(ii) *Page 22, Definition 2.1.3*
Let us interpret this for the case in which $M_1 = M_2 = \mathbf{R}$, with the Euclidean metric. $f: \mathbf{R} \to \mathbf{R}$ is continuous at a in $\mathbf{R}$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. This is precisely continuity at a in the familiar sense, as formulated in **S** Definition 1.4.3′ and discussed in Section 2.4 of *Unit 2*. ▲

EXAMPLE 5 We are able at once to recognize a number of general classes of continuous functions.

(a) If $\{A, d\}$ is a metric space, then the identity mapping $id_A: A \to A$ (that is, $id_A: x \mapsto x, \quad x \in A$) is continuous. More generally, if $H \subset A$ and $i: H \to A$ is the inclusion of H in A ($i(x) = x, \quad x \in H$), then i is (d_H, d)-continuous where d_H is the metric on H induced by d. Loosely expressed, the inclusion of a subspace in a metric space is continuous.

To see this, let $\varepsilon > 0$ be given. If $x, a \in H$ and $d_H(x, a) < \varepsilon$, then $d(i(x), i(a)) = d(x, a) = d_H(x, a) < \varepsilon$. Thus for each a in H and each given $\varepsilon > 0$ the condition of Definition 2.1.3 is satisfied by taking $\delta = \varepsilon$.

(b) If $\{A_1, d_1\}$ and $\{A_2, d_2\}$ are metric spaces, $f: A_1 \to A_2$ is a continuous function, and H_1, H_2 are subsets of A_1, A_2 respectively such that $f(H_1) \subset H_2$, then the restriction of f to H_1, $g: H_1 \to H_2$ given by putting $g(x) = f(x), x \in H_1$, is continuous with respect to the subspace metrics on H_1 and H_2.



Let $a \in H_1$ and let $\varepsilon > 0$ be given. Since f is continuous, there exists $\delta > 0$ such that $d_2(f(x), f(a)) < \varepsilon$ for every x in A_1 such that $d_1(x, a) < \delta$. Thus if $x \in H_1$ and $d_1(x, a) < \delta$, then $d_2(g(x), g(a)) = d_2(f(x), f(a)) < \varepsilon$. Therefore g is continuous at a for each a in H_1.

(c) If $\{A_1, d_1\}$ and $\{A_2, d_2\}$ are metric spaces and $f: A_1 \to A_2$ is a constant function (that is, $f(x) = c$ for all $x \in A_1$, where $c \in A_2$), then f is a continuous function. For let $a \in A_1$, let $\varepsilon > 0$ be given and choose $\delta > 0$ arbitrarily. If $x \in A_1$ and $d_1(x, a) < \delta$, then $d_2(f(x), f(a)) = d_2(c, c) = 0 < \varepsilon$. Therefore f is continuous at a for each a in A_1.

(d) If $M_1 = \{A_1, d_1\}$ and $M_2 = \{A_2, d_2\}$ are metric spaces and M_1 is discrete (S Example 2.2.2) then every map $f\colon A_1 \to A_2$ is (d_1, d_2)-continuous. For let $a \in A_1$ and let $\varepsilon > 0$ be given. Take $\delta = 1$. If $x \in A_1$ and $d_1(x, a) < \delta$, then $x = a$ so that $d_2(f(x), f(a)) = d_2(f(a), f(a)) = 0 < \varepsilon$. Therefore f is continuous at a for every a in A_1.

EXAMPLE 6 Consider the metric space $\{\mathbf{R}, e\}$ introduced in Example 1. Every function $f\colon \mathbf{R} \to \mathbf{R}$ is (e, e)-continuous at a in $\mathbf{R}$ if $a \neq 0$. To see this let $\varepsilon > 0$ be given and choose δ such that $0 < \delta \leq |a|$. If $e(x, a) < \delta$ then, as shown in Example 1, $x = a$ so that $f(x) = f(a)$ and hence $e(f(x), f(a)) = 0 < \varepsilon$.

Now let us consider two particular functions. If $g\colon \mathbf{R} \to \mathbf{R}$ is given by $g(x) = -x$, $x \in \mathbf{R}$, then g is an (e, e)-continuous function. In view of the above, we need only show that g is (e, e)-continuous at 0.

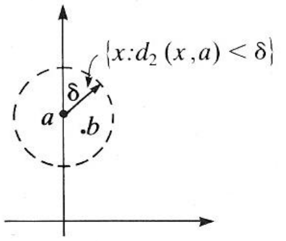
Let $\varepsilon > 0$ be given. Since $e(x, 0) = |x|$, if $e(x, 0) < \varepsilon$ then

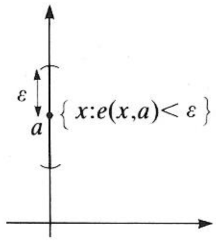
$$e(g(x), g(0)) = e(-x, 0) = |-x| = |x| < \varepsilon.$$

Thus if we take $\delta = \varepsilon$ then $e(x, 0) < \delta$ implies that $e(g(x), g(0)) < \varepsilon$. Hence g is (e, e)-continuous at 0.

Finally, consider $h\colon \mathbf{R} \to \mathbf{R}$ given by $h(x) = 1 + x$, $x \in \mathbf{R}$. This function is not (e, e)-continuous at 0. To see this, suppose, on the contrary, that h is (e, e)-continuous at 0. Then there exists $\delta > 0$ such that $e(x, 0) < \delta$ implies that $e(h(x), h(0)) < 1$ (taking $\varepsilon = 1$ in Definition 2.1.3). But $h(0) = 1$ so that $e(h(x), h(0)) < 1$ if and only if $h(x) = 1$, i.e. if and only if $x = 0$. Since $e(x, 0) < \delta$ if and only if $|x| < \delta$, this shows that $|x| < \delta$ if and only if $x = 0$; and so we have a contradiction. Thus h is not (e, e)-continuous at 0.

EXAMPLE 7 Let e be the metric on $\mathbf{R}^2$ introduced in Example 2 and let $i\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the identity mapping on $\mathbf{R}^2$ ($i(x) = x, x \in \mathbf{R}^2$). You will recall that points which are close in the Euclidean sense may be far apart with respect to the metric e. Thus we should suspect that i is not (d_2, e)-continuous, where d_2 is the Euclidean metric on $\mathbf{R}^2$; let us prove it. Suppose, in fact, that i is (d_2, e)-continuous at $a = (0, t)$ where $t \neq 0$. Let ε be a real number such that $0 < \varepsilon \leq |t|$.





By (d_2, e)-continuity at a, there exists $\delta > 0$ such that $d_2(x, a) < \delta$ implies that $e(i(x), i(a)) = e(x, a) < \varepsilon$. But if $x = (x_1, x_2) \in \mathbf{R}^2$ and $x_1 \neq 0$ then $e(x, a) > d_2(\mathbf{0}, a) = |t| \geq \varepsilon$. Thus if $x = (x_1, x_2)$ and $e(x, a) < \varepsilon$ then $x_1 = 0$. It follows that if $x = (x_1, x_2)$ and $d_2(x, a) < \delta$ then $x_1 = 0$. This is a contradiction. For example, if $b = (\frac{1}{2}\delta, t)$ then $d_2(a, b) = \frac{1}{2}\delta < \delta$ and $\frac{1}{2}\delta \neq 0$. Thus i is not (d_2, e)-continuous.

EXAMPLE 8 Let $\{A, d\}$, $\{A', d'\}$ and $\{X, e\}$ be metric spaces and let $a \in X$. Let $f\colon X \to A$ be (e, d)-continuous at a and let $g\colon X \to A'$ be (e, d')-continuous at a. In *Unit 1*, we considered the function

$$h\colon X \to A \times A'$$

given by $h(x) = (f(x), g(x))$, $x \in X$. We shall show that h is (e, d_∞)-continuous at a where d_∞ is the metric on $A \times A'$ given by

$$d_\infty((x_1, x_2), (y_1, y_2)) = \operatorname{Max}\{d(x_1, y_1), d'(x_2, y_2)\}.$$

Let $\varepsilon > 0$ be given. Since f is (e, d)-continuous at a, there exists $\delta_1 > 0$ such that $e(x, a) < \delta_1$ implies that $d(f(x), f(a)) < \varepsilon$. Also, since g is (e, d')-continuous at a, there exists $\delta_2 > 0$ such that $e(x, a) < \delta_2$ implies that $d'(g(x), g(a)) < \varepsilon$. Since $h(x) = (f(x), g(x))$ if $x \in X$, it follows that

$$d_\infty(h(x), h(a)) = \text{Max}\ \{d(f(x), f(a)), d'(g(x), g(a))\}.$$

Let $\delta = \text{Min}\ \{\delta_1, \delta_2\}$. Then $\delta > 0$ and if $e(x, a) < \delta$, then both $d(f(x), f(a)) < \varepsilon$ and $d'(g(x), g(a)) < \varepsilon$ so that $d_\infty(h(x), h(a)) < \varepsilon$. Therefore h is continuous at a.

SAQ 8 Show that $f: (0, 1] \to (0, 1]$ given by $f(x) = x^2$, $x \in (0, 1]$, is continuous if $(0, 1]$ has the Euclidean metric.

SAQ 9 Let $f: \mathbf{R} \to S^1$ be given by

$$f(t) = (\cos t, \sin t), \quad t \in \mathbf{R}.$$

Show that f is continuous if $\mathbf{R}$ has the Euclidean metric and S^1 has the metric induced by the metric d_∞ on $\mathbf{R}^2$. [Hint: Use Example 8 and Example 5 part (b).]

SAQ 10 Let $g = f|[0, 2\pi)$ where f is the function introduced in SAQ 9. Show that if $[0, 2\pi)$ has the Euclidean metric and S^1 has the metric induced by d_∞ then

$$g: [0, 2\pi) \to S^1$$

is a continuous bijection but g^{-1} is not continuous.

SAQ 11 Let $\{A, d\}$ be a metric space and let $c \in A$. Show that the function $f: A \to \mathbf{R}$ defined by

$$f: x \mapsto d(x, c), \quad x \in A,$$

is continuous if A has the metric d and $\mathbf{R}$ has the Euclidean metric. [Hint: Recall the result of SAQ 4.]

SOLUTION TO SAQ 8 We must show that f is continuous at c for each c in $(0, 1]$. If $0 < c < 1$ then the argument is essentially the same as that given in Example 9 of *Unit 2* Section 2.4. Thus it remains to show that f is continuous at 1. We must prove that if $\varepsilon > 0$ is given, then there exists $\delta > 0$ such that if $x \in (0, 1]$ and $1 - \delta < x \leq 1$ then $|f(x) - f(1)| = 1 - x^2 < \varepsilon$. Clearly there is no loss of generality in supposing that $\varepsilon < 1$.

Now

$$1 - x^2 = (1 + x)(1 - x) < 2(1 - x) \qquad \text{if } x \in (0, 1].$$

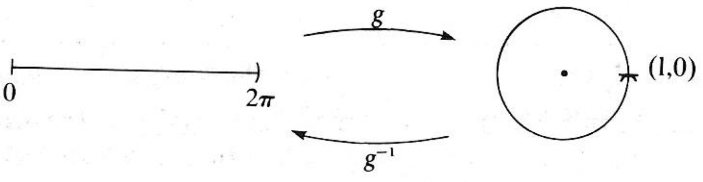
Also if $1 - \delta < x$ then $1 - x < \delta$. Thus if we take $\delta = \dfrac{\varepsilon}{2}$ then $1 - \delta < x \leq 1$ implies that $|f(x) - f(1)| < \varepsilon$. Hence f is continuous at 1. [An alternative argument could be based on Example 5, part (b).] ■

SOLUTION TO SAQ 9 Since the functions

$$t \mapsto \cos t, \quad t \mapsto \sin t, \qquad t \in \mathbf{R},$$

are continuous, it follows from the result of Example 7 that $h: \mathbf{R} \to \mathbf{R}^2$ given by $h(t) = (\cos t, \sin t)$, $t \in \mathbf{R}$, is continuous if $\mathbf{R}^2$ has the metric d_∞. Since $h(\mathbf{R}) = S^1$ and $f(t) = h(t)$, $t \in \mathbf{R}$, it follows from Example 5, part (b), that f is continuous. ■

SOLUTION TO SAQ 10 It follows from Example 5, part (b), that $g: [0, 2\pi) \to S^1$ is continuous.



It is easy to see that g is a bijection. Very loosely, we can appreciate that g^{-1} is not continuous since points near $(1,0)$ on S^1 can map to opposite ends of $[0,2\pi)$. To establish this carefully, let $a=(1,0)\in S^1$ and suppose that $h=g^{-1}$ is continuous at a. Then, taking $\varepsilon=\pi$ in the condition for continuity at a, there exists $\delta>0$ such that $|h(x)-h(a)|<\pi$ if $x\in S^1$ and $d_\infty(x,a)<\delta$. Since $h(a)=0$ it follows that $0\le h(x)<\pi$ for every x in S^1 such that $d_\infty(x,a)<\delta$.

Let $b=(b_1,-\frac{1}{2}\delta)$ where $b_1^2=1-\frac{1}{4}\delta^2$, so that $b\in S^1$ and $d_\infty(b,a)<\delta$. Then $0\le h(b)<\pi$. But $g(h(b))=b$ so that $\sin h(b)=-\frac{1}{2}\delta<0$. This contradiction shows that h is not continuous at a. ■

SOLUTION TO SAQ 11

Let us put $f(x)=d(x,c)$, $x\in A$. Let $a\in A$ and let $\varepsilon>0$ be given. Then

$$|f(x)-f(a)|=|d(x,c)-d(a,c)|\le d(x,a)$$

by the result of SAQ 4. If we take $\delta=\varepsilon$ then $d(x,a)<\delta$ implies that $|f(x)-f(a)|<\varepsilon$. Thus f is continuous at a for each a in A. ■

3.4 Further Examples

The metric spaces considered in this section have important applications in advanced analysis. Do not be discouraged if you find it hard going. We should like you to study it enough to make acceptable the idea that there are interesting metric spaces whose points are functions or sequences.

READ S page 26, line 12 to page 32, line 2.

NOTES

(i) *Page 28, line 11*
We shall use $\mathscr{C}[a,b]$ to denote the set of continuous real functions on the bounded closed interval $[a,b]$. When the context shows that $\mathscr{C}[a,b]$ is to be regarded as a metric space, and the metric is not specified, it should be assumed that it has the sup metric, subsequently called d_∞.

(ii) *Page 29, line −9*
From **S** Examples 2.2.8, 2.2.9 and 2.2.11 we obtain three metrics, d_∞, d_1 and d_2, on the set $\mathscr{C}[a,b]$. If $f,g\in\mathscr{C}[a,b]$ then

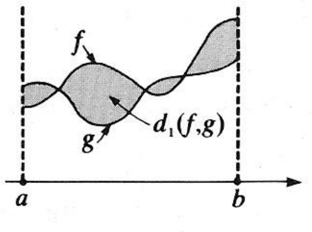
$$d_1(f,g)=\int_a^b|f(x)-g(x)|dx,$$

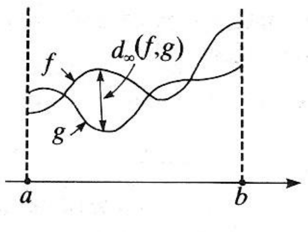
$$d_2(f,g)=\left\{\int_a^b(f(x)-g(x))^2dx\right\}^{\frac{1}{2}}$$

and

$$d_\infty(f,g)=\sup_{x\in[a,b]}|f(x)-g(x)|.$$

There is a sense in which the notation is consistent with that introduced in **S** Example 2.2.3 for Euclidean spaces. The metrics d_1 and d_∞ have a nice geometrical interpretation.

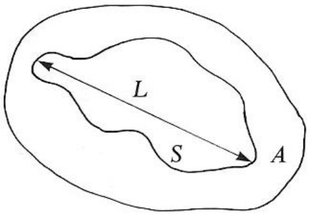




If the graphs of f and g are as shown, then $d_1(f,g)$ is the area of the region shaded in the sketch. Also, $d_\infty(f,g)$ is the maximum value of the difference of the ordinates $f(x)$ and $g(x)$ at x, for x in $[a,b]$.

(iii) *Page 28, Lemma 2.2.10*
We suggest that you accept Lemma 2.2.10 and the Cauchy–Schwarz–Bunyakovsky inequality, without worrying about their proofs.

(iv) *Page 29, Definition 2.2.12*
We note the following characterization of boundedness: a non-empty subset S of a metric space $\{A, d\}$ is bounded if and only if there exists a real number L such that $d(x, y) \leq L$ for all x, y in S.



For if this condition is satisfied, take a fixed a in S; then $d(x, a) \leq L$ for all x in S so that S is a bounded set. Conversely, if S is a bounded set then there exist a in A and a real number K such that $d(x, a) \leq K$ for all x in S. If $x, y \in S$ then

$$d(x, y) \leq d(x, a) + d(y, a) \leq 2K$$

and so the stated condition is satisfied. It follows that if S is a non-empty bounded subset of a metric space $\{A, d\}$, then the set D of real numbers of the form $d(x, y)$ for x, y in S is non-empty and bounded above, so that $\sup D$ exists (*Unit 2* Section 2.1), justifying the definition of diam S (S Definition 2.2.13).

(v) *Page 30, Example 2.2.16*
Intuitively, the Hilbert sequence space ℓ^2 is a Euclidean space of infinitely many dimensions. For each $x = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$, put

$$\hat{x} = (x_1, \ldots, x_n, 0, 0, 0, \ldots).$$

Then $\hat{x}$ is a point of ℓ^2 and $x \mapsto \hat{x}$, $x \in \mathbf{R}^n$, is an injection of $\mathbf{R}^n$ into ℓ^2. Also, if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are points of $\mathbf{R}^n$ then, in ℓ^2,

$$d_2(\hat{x}, \hat{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

so that $d_2(\hat{x}, \hat{y}) = d_2(x, y)$, the Euclidean distance between x and y in $\mathbf{R}^n$. It is tempting to regard the subspace of ℓ^2 consisting of the points $\hat{x}$, for x in $\mathbf{R}^n$, as a copy of $\mathbf{R}^n$ and so to conclude that ℓ^2 contains a copy of the Euclidean space $\mathbf{R}^n$ for every positive integer n. You will be able to make this idea more precise after studying Section 3.6. ▲

NOW START THE TAPE

EXAMPLE 9 We show that if k is a real number, then the function $T\colon \mathscr{C}[a, b] \to \mathscr{C}[a, b]$ given by

$$T(f) = kf, \quad f \in \mathscr{C}[a, b],$$

is (d_∞, d_1)-continuous.

First we note that if $f \in \mathscr{C}[a, b]$ then $T(f)$ is the continuous real function on $[a, b]$ given by $T(f)(x) = kf(x)$ for all $x \in [a, b]$, so that $Tf \in \mathscr{C}[a, b]$. If $k = 0$ then T is a constant mapping and so is continuous.

Next suppose that $k \neq 0$. Now if f, $g \in \mathscr{C}[a, b]$ then

$$d_1(T(f), T(g)) = \int_a^b |kf(x) - kg(x)|dx$$
$$= |k| \int_a^b |f(x) - g(x)|dx.$$

But $|f(x) - g(x)| \leq d_\infty(f, g)$ for all x in $[a, b]$ so that

$$\int_a^b |f(x) - g(x)|dx \leq (b - a)\, d_\infty(f, g),$$

and hence

$$d_1(T(f), T(g)) \leq |k|(b - a)\, d_\infty(f, g).$$

Now let $f \in \mathscr{C}[a, b]$ and let $\varepsilon > 0$ be given. If we take $\delta = \varepsilon/|k|(b - a)$ then $d_1(T(f), T(g)) < \varepsilon$ for every g in $\mathscr{C}[a, b]$ such that $d_\infty(f, g) < \delta$. Therefore T is continuous at f for each f in $\mathscr{C}[a, b]$.

SAQ 12 Let f and g be the elements of $\mathscr{C}[0, 1]$ given by

$$f(x) = 2x^2, \quad g(x) = x, \qquad x \in [0, 1]$$

and let d_∞, d_1 be the metrics on $\mathscr{C}[0, 1]$ defined as in **S** Examples 2.2.7 and 2.2.9 respectively. Find $d_\infty(f, g)$ and $d_1(f, g)$.

SAQ 13 Show that $\theta: \mathscr{C}[a, b] \to \mathbf{R}$ given by

$$\theta(f) = f(a), \quad f \in \mathscr{C}[a, b]$$

is continuous with respect to the sup metric d_∞ on $\mathscr{C}[a, b]$ and the Euclidean metric on $\mathbf{R}$.

SAQ 14 Prove that if B and C are bounded subsets of a metric space and $B \cap C \neq \varnothing$ then

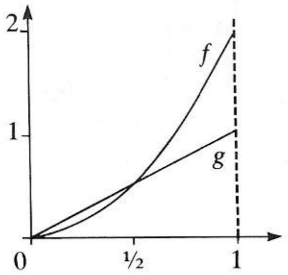
$$\operatorname{diam}(B \cup C) \leq \operatorname{diam} B + \operatorname{diam} C.$$

SAQ 15 Which of the following sequences are points of the Hilbert sequence space ℓ^2?

$$x = (1, 0, 0, \ldots), \quad y = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots\right), \quad z = \left(1, \frac{1}{2}, \frac{1}{2^2}, \ldots\right).$$

Find the distance between those sequences which are points of ℓ^2.

SOLUTION TO SAQ 12 The graphs of f and g are sketched below.



Since

$$|f(x) - g(x)| = \begin{cases} x(1 - 2x), & \text{if } 0 \leq x \leq \frac{1}{2}, \\ x(2x - 1), & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

we see using the maximum/minimum method of the calculus that the maximum value of $|f(x) - g(x)|$ on $[0, 1]$ occurs when $x = 1$. Thus

$$d_\infty(f, g) = f(1) - g(1) = 1.$$

Also

$$d_1(f, g) = \int_0^1 |f(x) - g(x)|dx$$
$$= \int_0^{\frac{1}{2}} x(1 - 2x)dx + \int_{\frac{1}{2}}^1 x(2x - 1)dx = \tfrac{1}{4}. \quad \blacksquare$$

SOLUTION TO SAQ 13 Let $f \in \mathscr{C}[a,b]$; we shall show that θ is continuous at f. Let $\varepsilon > 0$ be given. If $g \in \mathscr{C}[a,b]$ then

$$|f(x) - g(x)| \leq d_\infty(f,g)$$

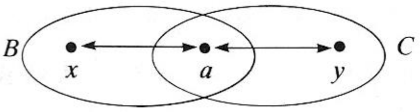
for every x in $[a,b]$ so that, in particular,

$$|\theta(f) - \theta(g)| = |f(a) - g(a)| \leq d_\infty(f,g).$$

Thus if $g \in \mathscr{C}[a,b]$ and $d_\infty(f,g) < \delta$, where $\delta = \varepsilon$, then $|\theta(f) - \theta(g)| < \varepsilon$. ■

SOLUTION TO SAQ 14 Let B and C be bounded subsets of $\{A,d\}$ and put $K = \text{diam } B$, $L = \text{diam } C$. We must show that $d(x,y) \leq K + L$ for all x, y in $B \cup C$. This clearly holds if x and y are both elements of B or if x and y are both elements of C. Thus we suppose that $x \in B$ and $y \in C$. Since $B \cap C \neq \varnothing$, there exists an element a of $B \cap C$. Then

$$d(x,y) \leq d(x,a) + d(a,y).$$



But x, $a \in B$ so that $d(x,a) \leq K$. Similarly, a, $y \in C$ so that $d(a,y) \leq L$. Therefore $d(x,y) \leq K + L$. ■

SOLUTION TO SAQ 15 Clearly $x \in \ell^2$. If we put $y = (y_i)$ then $y_i = \dfrac{1}{\sqrt{i}}$ so that

$$\sum_{i=1}^{\infty} y_i^2 = \sum_{i=1}^{\infty} \frac{1}{i}.$$

The harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$ is known to be divergent so that y is not a point of ℓ^2. If we put $z = (z_i)$ then $z_i = \dfrac{1}{2^{i-1}}$, so that

$$\sum_{i=1}^{\infty} z_i^2 = \sum_{i=1}^{\infty} \frac{1}{4^{i-1}},$$

which is a geometric series converging to $\frac{4}{3}$. Thus $z \in \ell^2$.

Finally,

$$(d_2(x,z))^2 = (1-1) + \frac{1}{4} + \frac{1}{4^2} + \cdots = \sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{1}{4}\left(1 - \frac{1}{4}\right)^{-1} = \frac{1}{3}.$$

Therefore $d_2(x,z) = \dfrac{1}{\sqrt{3}}$. ■

Further reading. If you are interested in the use of ℓ^2 in analysis, a good place to start is: *Functional Analysis*, W. Rudin (McGraw-Hill, 1973).

3.5 Open Balls and Open Sets

In our study of continuity of functions with domain or codomain a metric space $\{A,d\}$, we have frequently been concerned with sets of the form $\{x \in A : d(x,a) < \varepsilon\}$ where $a \in A$ and $\varepsilon > 0$. In this section we take a closer look at such sets, which are called *open balls*. This leads us to the important concept of an *open set* in a metric space and a new characterization of continuity.

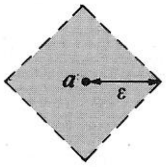
READ S all of Section 2.3, pages 33–38.

NOTES (i) *Page 33, Definition 2.3.1*

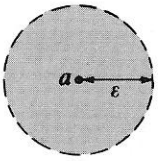
It will sometimes be convenient to say that the open ball $B_\varepsilon(a)$ has *centre a* and *radius* ε.

(ii) *Page 33, Example 2.3.3*

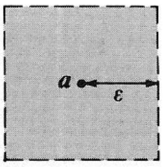
Here are sketches of the open balls $B_\varepsilon(a;\, d_1)$, $B_\varepsilon(a;\, d_2)$ and $B_\varepsilon(a;\, d_\infty)$ in $\mathbf{R}^2$.



$B_\varepsilon(a;d_1)$



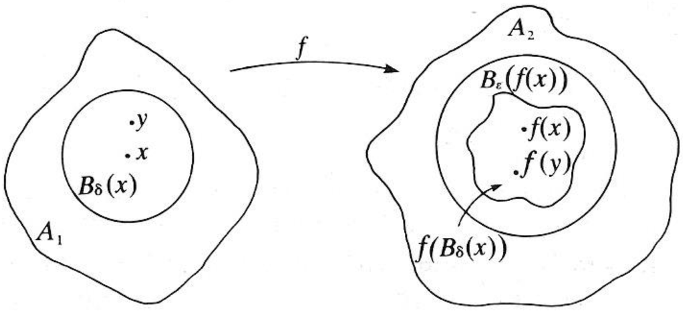
$B_\varepsilon(a;d_2)$



$B_\varepsilon(a;d_\infty)$

(iii) *Page 33, Definition 2.3.6*

We notice that **S** has made a subtle change in notation from Definition 2.1.3. Given metric spaces $M_1 = \{A_1, d_1\}$ and $M_2 = \{A_2, d_2\}$, the statement that $f\colon M_1 \to M_2$ is continuous at x is to be understood to mean that $f\colon A_1 \to A_2$ is a function which is (d_1, d_2)-continuous at x in A_1.



Now let us translate Definition 2.1.3 to the present form. Let $M_1 = \{A_1, d_1\}$, $M_2 = \{A_2, d_2\}$, let $f\colon A_1 \to A_2$ be a function and let $x \in A_1$. Let $\varepsilon > 0$ be given. Then, by Definition 2.1.3 there exists $\delta > 0$ such that

$$d_1(y, x) < \delta \quad \Rightarrow \quad d_2(f(y), f(x)) < \varepsilon$$

which we can rewrite in the form

$$y \in B_\delta(x, d_1) \quad \Rightarrow \quad f(y) \in B_\varepsilon(f(x); d_2).$$

Since this is equivalent to the assertion that

$$f(B_\delta(x; d_1)) \subset B_\varepsilon(f(x); d_2),$$

we have shown that continuity of f according to Definition 2.1.3 implies continuity according to Definition 2.3.6. As all the steps in this argument can be reversed, it follows that continuity according to Definition 2.3.6 implies continuity according to Definition 2.1.3. So the two definitions are equivalent.

(iv) *Page 35, Example 2.3.10*

To show that a subset H of M is *not* open in M we must find a point x of H such that for every $\varepsilon > 0$, $B_\varepsilon(x)$ contains some point not in H. Since $\varnothing$ has no points, it follows that $\varnothing$ is open in M.

(v) *Page 35, line −13*

A *singleton set* is a set with precisely one element.

(vi) *Page 36, Proposition 2.3.13*

Proposition 2.3.13 will be very important to us; it permits us to characterize continuity in terms of open sets instead of relations involving metrics. This enables us (in *Unit 4*) to define continuity of functions between topological spaces.

We should note that this proposition gives a necessary and sufficient condition for 'global' continuity, i.e. for continuity at every point of the domain. It will help your understanding of the statement and proof if you refresh your memory, from *Unit 1* Section 1.3, about the inverse image of a set with respect to a function: if $f\colon A_1 \to A_2$ is a function and $U \subset A_2$, then

$$f^{-1}(U) = \{x \in A_1 : f(x) \in U\}.$$

An important property of the inverse image, used twice in this proof, is that if $B \subset A_1$ then $B \subset f^{-1}(U)$ if and only if $f(B) \subset U$.

(vii) *Page 38, Propositions 2.3.17 and 2.3.19*

These propositions should be noted carefully. In the next unit we shall be studying more general objects called *topological spaces*. From these results, we shall be able to see that naturally associated with each metric space there is a topological space. The results of SAQ 18 and SAQ 19 below will also be significant in later work. ▲

EXAMPLE 10 Let us look again at the metric space $M = \{\mathbf{R}, e\}$ introduced in Example 1. In the notation we now have available, it was established that if $a \in \mathbf{R}$, $a \neq 0$, and $0 < r \leq |a|$ then

$$B_r(a) = \{a\},$$

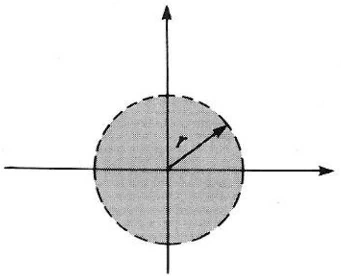
and also that, for each real number $r > 0$,

$$B_r(0) = \{x \in \mathbf{R}: |x| < r\} = (-r, r).$$

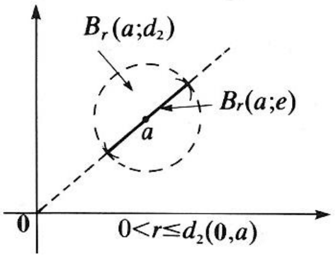
We can now identify the subsets which are open in M. If $V \subset \mathbf{R}$ and $0 \notin V$ then for each a in V, $B_\varepsilon(a) \subset V$ provided that $0 < \varepsilon \leq |a|$ so that V is open in M. Now suppose that $W \subset \mathbf{R}$ and $0 \in W$. If $b \in W$ and $b \neq 0$, there exists $\varepsilon > 0$ such that $B_\varepsilon(b) \subset W$, since $B_\varepsilon(b) = \{b\}$ if $0 < \varepsilon \leq |b|$. Since $B_\varepsilon(0) = (-\varepsilon, \varepsilon)$ if $\varepsilon > 0$, there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subset W$ if and only if there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset W$. Thus the sets open in M are all subsets V of $\mathbf{R}$ such that $0 \notin V$ together with all subsets V of $\mathbf{R}$ such that $0 \in V$ and there exists some $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset V$.

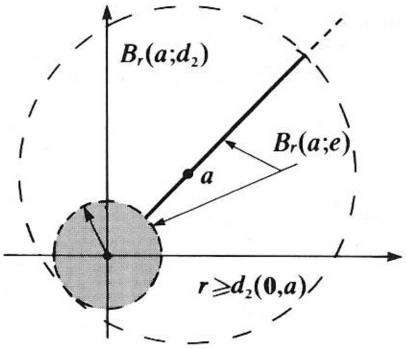
As an application of **S** Proposition 2.3.13, let us show that the identity map $i\colon \mathbf{R} \to \mathbf{R}$ ($i(x) = x, x \in \mathbf{R}$) is (e, d)-continuous, where d is the Euclidean metric for $\mathbf{R}$. Let V be a d-open subset of $\mathbf{R}$. Then $i^{-1}(V) = V$. If $0 \notin V$ then V is e-open. If $0 \in V$ then since V is d-open, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset V$ and so V is e-open. Hence i is (e, d)-continuous.

EXAMPLE 11 For the metric space $M = \{\mathbf{R}^2, e\}$, where e is the metric on $\mathbf{R}^2$ introduced in Example 2, let us find some typical open balls. Since $e(\mathbf{0}, x) = d_2(\mathbf{0}, x)$ for all x, where d_2 is the Euclidean metric, it follows that, for each $r > 0$, the open r-ball neighbourhood $B_r(\mathbf{0})$ of $\mathbf{0}$ in M is the interior of the (Euclidean) disc of radius r centred on $\mathbf{0}$.



If $a \in \mathbf{R}^2$ and $a \neq \mathbf{0}$, the nature of $B_r(a)$ depends on r. If $0 < r \leq d_2(\mathbf{0}, a)$ then $e(x, a) < r$ if and only if x lies on the line joining a to $\mathbf{0}$ and $d_2(x, a) < r$. If $r > d_2(\mathbf{0}, a)$ and we put $t = r - d_2(\mathbf{0}, a)$ then $e(x, a) < r$ if and only if either (i) x lies on the line joining a to $\mathbf{0}$ and $d_2(x, a) < r$ or (ii) $d_2(\mathbf{0}, x) < t$. Typical balls $B_r(a)$ are sketched on the next page.





This metric e is sometimes called the 'lollipop metric' on $\mathbf{R}^2$!

As an application of S Definition 2.3.6 let us show that the identity i: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ is (e, d_2)-continuous. (Recall that in Example 7 it was shown that i is not (d_2, e)-continuous.) We must show that for a in $\mathbf{R}^2$ and any given $\varepsilon > 0$, there exists $\delta > 0$ such that $B_\delta(a; e) \subset B_\varepsilon(a; d_2)$. But the discussion above shows that

$$B_\varepsilon(a; e) \subset B_\varepsilon(a; d_2)$$

for each a in $\mathbf{R}^2$ and each $\varepsilon > 0$.

EXAMPLE 12 Let $M = \{A, d\}$ be a metric space and let f: $A \rightarrow \mathbf{R}$ be a function. As an application of S Proposition 2.3.13 we can obtain a useful criterion for continuity of f (when $\mathbf{R}$ has the Euclidean metric): The function f is continuous if and only if the sets

$$V_t = \{x \in A : f(x) < t\} \quad \text{and} \quad W_t = \{x \in A : f(x) > t\}$$

are open in M for each t in $\mathbf{R}$. Since

$$V_t = f^{-1}((-\infty, t)) \quad \text{and} \quad W_t = f^{-1}((t, \infty)),$$

and $(-\infty, t)$, (t, ∞) are open in $\mathbf{R}$, we see that if f is continuous then V_t and W_t are open in M for every t. Conversely, suppose that V_t and W_t are open in M for every t. Let $a \in A$ and let $\varepsilon > 0$ be given. If $x \in V_{f(a)+\varepsilon}$ then $f(x) < f(a) + \varepsilon$ and if $x \in W_{f(a)-\varepsilon}$ then $f(x) > f(a) - \varepsilon$. Thus if $x \in G = V_{f(a)+\varepsilon} \cap W_{f(a)-\varepsilon}$ then

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon.$$

Now $V_{f(a)+\varepsilon}$ and $W_{f(a)-\varepsilon}$ are open in M (given), so that G is open in M by S Proposition 2.3.17. But $a \in G$ so there exists $\delta > 0$ such that $B_\delta(a) \subset G$. Thus if $d(x, a) < \delta$ then $x \in G$ so that

$$|f(x) - f(a)| < \varepsilon.$$

Hence f is continuous at a.

SAQ 16 For the metric space $\{\mathbf{R}^2, e\}$ introduced in Example 2, sketch the open balls $B_{\frac{1}{2}}(a)$ and $B_{\frac{3}{2}}(a)$, where $a = (\frac{3}{5}, -\frac{4}{5})$.

SAQ 17 In $\mathbf{R}$ with its usual Euclidean metric, which of the following subsets are open?

(i) $\mathbf{R} - \mathbf{Z}$
(ii) $\mathbf{R} - \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$
(iii) $\mathbf{R} - \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$
(iv) $\{x \in (0,1)\colon x \text{ is rational}\} = (0,1) \cap \mathbf{Q}$

SAQ 18 Show that a subset of a metric space M is open in M if and only if it is a union of open balls.

SAQ 19 Show that if x and y are distinct points of a metric space M, then there are disjoint sets V and W open in M such that $x \in V$ and $y \in W$.

SAQ 20 Use **S** Proposition 2.3.13 to show that the following functions are not continuous.

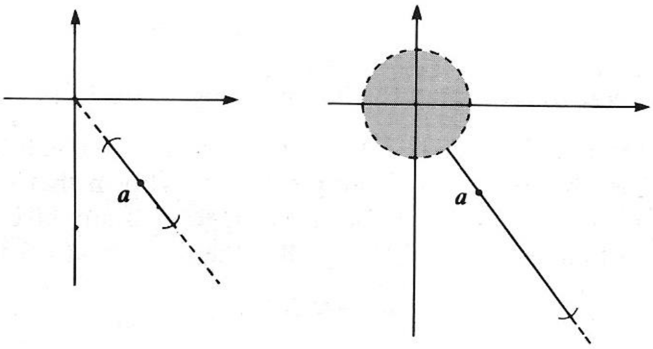
(i) $f: \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} x, & \text{if } x < 0, \\ 1 + x, & \text{if } x \geq 0. \end{cases}$$

(ii) $g: [0,1] \to [0,1]$ given by

$$g(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 1 - x, & \text{if } x \text{ is irrational.} \end{cases}$$

SOLUTION TO SAQ 16



■

SOLUTION TO SAQ 17

(i) $\mathbf{R} - \mathbf{Z} = \bigcup_{n \in \mathbf{Z}} (n, n+1)$ so that $\mathbf{R} - \mathbf{Z}$ is open by **S** Proposition 2.3.19.

(ii) Let $V = \mathbf{R} - \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Then $0 \in V$, but for each $\varepsilon > 0$ there exists some positive integer n such that $\frac{1}{n} < \varepsilon$. Thus there is no $\varepsilon > 0$ such that $B_\varepsilon(0) \subset V$. Therefore V is not open in **R**.

(iii) Let $W = \mathbf{R} - \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. If $x \in W$ then either $x < 0$ or $x > 1$ or $x \in \left(\frac{1}{n}, \frac{1}{n+1}\right)$ for some positive integer n. Thus

$$W = (-\infty, 0) \cup \left(\bigcup_{n \in \mathbf{N}} \left(\frac{1}{n}, \frac{1}{n+1} \right) \right) \cup (1, \infty).$$

Hence W is open in **R** by **S** Proposition 2.3.19.

(iv) Let $X = (0,1) \cap \mathbf{Q}$. If $a \in X$ and $\varepsilon > 0$ then $(a - \varepsilon, a + \varepsilon)$ contains an irrational number (irrational density property, *Unit 2* Section 2.1). Thus there is no $\varepsilon > 0$ such that $B_\varepsilon(a) \subset X$. Therefore X is not open in **R**. ■

SOLUTION TO SAQ 18 By **S** Proposition 2.3.19, a union of open balls of a metric space M is open in M. Conversely, let V be open in a metric space M. For each x in V there exists $\varepsilon_x > 0$ such that $x \in B_{\varepsilon_x}(x) \subset V$.

Thus $V = \bigcup_{x \in V} B_{\varepsilon_x}(x)$. ■

SOLUTION TO SAQ 19 If x and y are distinct then $d(x,y) > 0$. Let $\varepsilon = \frac{1}{2} d(x,y)$. If $V = B_\varepsilon(x)$ and $W = B_\varepsilon(y)$ then V and W are open in M and $x \in V$, $y \in W$. Finally, suppose there exists z such that $z \in V \cap W$. Then $d(x,z) < \varepsilon$ and $d(y,z) < \varepsilon$ so that

$$d(x,y) \leq d(x,z) + d(y,z) < 2\varepsilon = d(x,y).$$

This contradiction shows that V and W are disjoint. ■

SOLUTION TO SAQ 20

(i) Since $f^{-1}((0,\infty)) = [0,\infty)$ and $[0,\infty)$ is not open in **R**, it follows that f is not continuous.

(ii) The interval $[0, \frac{1}{2})$ is open in $[0,1]$ but

$$g^{-1}([0, \tfrac{1}{2})) = \{x \in [0, \tfrac{1}{2}): x \text{ is rational}\} \cup \{x \in (\tfrac{1}{2}, 1): x \text{ is irrational}\},$$

and this set is not open in $[0,1]$. Thus g is not continuous. ■

3.6 Equivalent Metrics

In this section we consider some equivalence relations on metrics on a given set, the most important of which is *topological equivalence*.

READ S from the beginning of Section 2.4, page 38, line −3, to the end of the chapter.

NOTES (i) *Page 39, line 16*

If $M = \{A, d\}$ is a metric space and $U \subset A$, then U is d-open means that U is open in M.

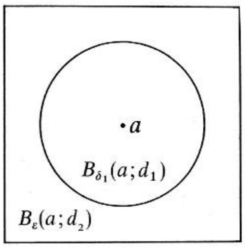
(ii) *Page 39, Definition 2.4.1*

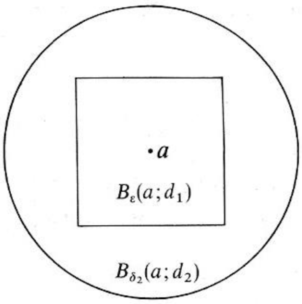
To say that two metrics d_1 and d_2 on a set A are topologically equivalent means that for every metric space $\{B, d\}$, a function $f\colon B \to A$ is (d, d_1)-continuous if and only if it is (d, d_2)-continuous, and also for every metric space $\{C, d'\}$, a function $f\colon A \to C$ is (d_1, d')-continuous if and only if it is (d_2, d')-continuous. This raises the question of how one would test whether or not metrics d_1 and d_2 on a set A are topologically equivalent, for it will be clear that this unwieldy definition cannot be applied directly. S Proposition 2.4.2 gives a more workable characterization. We now describe another equivalent characterization of topologically equivalent metrics. We shall prove that metrics d_1 and d_2 on a set A are topologically equivalent if and only if the following condition is satisfied:

(TE) *For each a in A and each $\varepsilon > 0$ there exists*

$\delta_1 > 0$ and $\delta_2 > 0$ such that

$B_{\delta_1}(a; d_1) \subset B_\varepsilon(a; d_2)$ and $B_{\delta_2}(a; d_2) \subset B_\varepsilon(a; d_1)$.





To prove this, suppose first that d_1 and d_2 are topologically equivalent. Then for each a in A and each $\varepsilon > 0$, the open ball $B_\varepsilon(a; d_2)$ is d_2-open and hence d_1-open, by S Proposition 2.4.2. But $a \in B_\varepsilon(a; d_2)$ and so there exists $\delta_1 > 0$ such that $B_{\delta_1}(a; d_1) \subset B_\varepsilon(a; d_2)$. The proof that the other half of condition (TE) holds is similar.

Conversely, suppose that (TE) holds and that U is a d_2-open set. If $a \in U$ then there exists $\varepsilon > 0$ such that $B_\varepsilon(a; d_2) \subset U$. By hypothesis, there exists $\delta_1 > 0$ such that $B_{\delta_1}(a; d_1) \subset B_\varepsilon(a; d_2) \subset U$. Hence U is d_1-open. The proof that each d_1-open set is d_2-open is similar. Hence, if (TE) holds then d_1 and d_2 are topologically equivalent by S Proposition 2.4.2.

We now have another means of testing for topological equivalence. It is applied in Example 13 and you should apply it in SAQ 21.

(iii) *Page 40, Example 2.4.5*

We verify the chain of inequalities which establishes the Lipschitz equivalence of the metrics d_1, d_2 and d_∞ on $\mathbf{R}^n$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ and let us put $|x_i - y_i| = a_i$ for $i = 1, \ldots, n$ and $d_1(x, y) = a$, $d_2(x, y) = b$ and $d_\infty(x, y) = c$. We shall prove that

$$a \geq b \geq c \geq n^{-\frac{1}{2}} b \geq n^{-1} a.$$

Since $a = a_1 + \cdots + a_n$ we have

$$a^2 = a_1^2 + \cdots + a_n^2 + 2(a_1 a_2 + \cdots + a_{n-1} a_n)$$

so that

$$a^2 \geq a_1^2 + \cdots + a_n^2 = b^2$$

and hence $a \geq b$ since $a, b \geq 0$. Next we note that $c = a_j$ for some $j = 1, \ldots, n$, so that $c^2 \leq b^2$ and hence $b \geq c$ (since $c \geq 0$). Since $a_i \leq c$ for $i = 1, \ldots, n$ it follows that $b^2 \leq nc^2$ so that $c \geq n^{-\frac{1}{2}} b$. Finally, by Cauchy's inequality (**S** page 22),

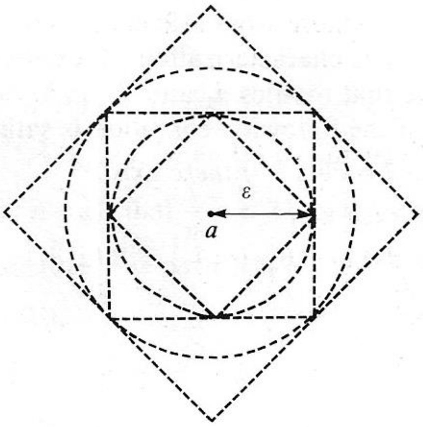
$$n(a_1^2 + \cdots + a_n^2) \geq (a_1 + \cdots + a_n)^2$$

so that $nb^2 \geq a^2$ and hence $a \leq n^{\frac{1}{2}} b$.

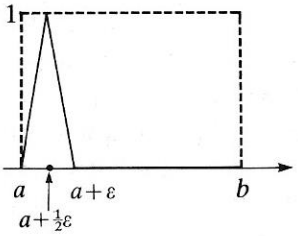
Let us look at the geometrical interpretation of this chain of inequalities in the case $n = 2$. If $a \in \mathbf{R}^2$ and ε is a positive real number then

$$B_\varepsilon(a; d_1) \subset B_\varepsilon(a; d_2) \subset B_\varepsilon(a; d_\infty) \subset B_{\sqrt{2}\varepsilon}(a; d_2) \subset B_{2\varepsilon}(a; d_1).$$

Bearing in mind the descriptions of the open balls in Note (ii) of the preceding section, we have the following picture.



(iv) *Page 41, line 7*
A continuous function f on $[a, b]$, such that $d_1(f, 0) < \varepsilon$ and $d_\infty(f, 0) = 1$, can be obtained by taking a function whose graph has the form shown below;



or, more formally, by putting

$$f(x) = \begin{cases} 2\varepsilon^{-1}(x - a), & \text{if } a \leq x < a + \frac{1}{2}\varepsilon, \\ 2\varepsilon^{-1}(a + \varepsilon - x), & \text{if } a + \frac{1}{2}\varepsilon \leq x < a + \varepsilon, \\ 0, & \text{if } a + \varepsilon \leq x \leq b. \end{cases}$$

(Without loss of generality, we have assumed that $\varepsilon < (b - a)$.) Then $d_1(f, 0) = 1 \times \frac{1}{2}\varepsilon = \frac{1}{2}\varepsilon < \varepsilon$ whilst $d_\infty(f, 0) = f(a + \frac{1}{2}\varepsilon) = 1$.

(v) *Page 41, Definition 2.4.7*
The condition that f^{-1} is continuous cannot be omitted. In SAQ 10 we had an example of a continuous one–one correspondence g such that g^{-1} is not continuous.

(vi) *Page 41, Definition 2.4.8*
It is clear from the definition that if $f: M \to M'$ is a Lipschitz equivalence and one of M and M' is bounded then so is the other.

(vii) *Page 41, line −12*
As S mentions, each Lipschitz equivalence is a topological equivalence, and it is clear that an isometry is a Lipschitz equivalence. The identity mapping of $(\mathbf{R}^n, d_1)$ to $(\mathbf{R}^n, d_2)$ is an example of a Lipschitz equivalence which is not an isometry. Example 13 below provides us with an example of a topological equivalence which is not a Lipschitz equivalence. ▲

EXAMPLE 13 Let $\{A, d\}$ be a metric space. In SAQ 3 you found that there is a metric $\overline{d}$ on A given by

$$\overline{d}(x,y) = \text{Min}\{1, d(x,y)\}, \quad (x,y) \in A \times A.$$

We now show that the metrics d and $\overline{d}$ are topologically equivalent. Intuitively, one would expect this to be true since the metric spaces $\{A, d\}$ and $\{A, \overline{d}\}$ have the same 'small' open balls. Let $a \in A$ and let $\varepsilon > 0$ be given. Since $\overline{d}(a,x) \leq d(a,x)$ for all x in A, we see that $B_\varepsilon(a; d) \subset B_\varepsilon(a; \overline{d})$. Now let $\delta = \text{Min}\{1, \varepsilon\}$. If $\overline{d}(a,x) < \delta$ then $\overline{d}(a,x) < 1$ so that $\overline{d}(a,x) = d(a,x)$ and hence $d(a,x) < \delta \leq \varepsilon$. Hence d and $\overline{d}$ are topologically equivalent, by the condition (TE) of Note (ii) above. We note that $\{A, \overline{d}\}$ is bounded since $\overline{d}(x,y) \leq 1$ for all x, y in A. So *any* metric space $\{A, d\}$ is topologically equivalent to a *bounded* metric space, namely $\{A, \overline{d}\}$.

For example, the space $\{\mathbf{R}, d\}$, where d is the Euclidean metric, is not bounded but is topologically equivalent to the space $\{\mathbf{R}, \overline{d}\}$, which is bounded. Thus boundedness is not preserved by topological equivalence.

It follows from Note (vi) above that the identity mapping of $\mathbf{R}$ with the Euclidean metric d to $\mathbf{R}$ with the associated metric $\overline{d}$ is a topological equivalence which is not a Lipschitz equivalence.

SAQ 21 Prove that the function e: $(0,1) \times (0,1) \to \mathbf{R}$ given by

$$e(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \quad x, y \in (0,1),$$

is a metric on $(0,1)$ which is topologically equivalent but not Lipschitz equivalent to the Euclidean metric.

SAQ 22 Let f: $(0,1) \to (a,b)$ be given by

$$f(t) = a + t(b-a), \quad t \in (0,1).$$

Show that if the intervals have the Euclidean metric then f is a Lipschitz equivalence. When is f an isometry?

SAQ 23 Show that if $\mathbf{R}$ has the Euclidean metric then $f\colon \mathbf{R} \to (-1,1)$ given by

$$f(x) = \frac{x}{1+|x|}, \quad x \in \mathbf{R},$$

is a topological equivalence of $\mathbf{R}$ with its subspace $(-1,1)$.

SOLUTION TO SAQ 21 It is clear that (M1) and (M2) are satisfied. If x, y, $z \in (0,1)$ then

$$\frac{1}{x} - \frac{1}{z} = \left(\frac{1}{x} - \frac{1}{y}\right) + \left(\frac{1}{y} - \frac{1}{z}\right)$$

so that

$$e(x,z) \leq e(x,y) + e(y,z)$$

by the triangle inequality for the modulus function. Thus e is a metric on $(0,1)$. Let d denote the Euclidean metric on $(0,1)$. Let $a \in (0,1)$ and let $\varepsilon > 0$ be given. Since

$$e(x,a) = \frac{|x-a|}{xa}$$

and $x < 1$, $a < 1$ it follows that $e(x,a) > d(x,a)$. Thus $B_\varepsilon(a; e) \subset B_\varepsilon(a; d)$.

Also, if $|x - a| < \frac{1}{2}a$ then $x > \frac{1}{2}a$ so that

$$e(x, a) < \frac{2}{a^2}|x - a| = \frac{2}{a^2}d(x, a).$$

Thus if $\delta = \text{Min}\ \{\frac{1}{2}a, \frac{1}{2}a^2\varepsilon\}$ then $d(x, a) < \delta$ implies that $e(x, a) < \varepsilon$ and hence $B_\delta(a; d) \subset B_\varepsilon(a; e)$. Therefore d and e are topologically equivalent.

Since $e\left(\frac{1}{n}, \frac{1}{2}\right) = n - 2$ for each n, we see that $\{(0, 1), e\}$ is not bounded, whereas $\{(0, 1), d\}$ is bounded. Hence d and e are not Lipschitz equivalent. ■

SOLUTION TO SAQ 22 Clearly f is a bijection. If $s, t \in (0, 1)$ then

$$|f(s) - f(t)| = (b - a)|s - t|$$

so that f is a Lipschitz equivalence. Also f is an isometry if and only if $b - a = 1$. ■

SOLUTION TO SAQ 23 Since $1 + |x| > 0$ if $x \in \mathbf{R}$, the function f is continuous (SAQ 20 parts (iv) and (v), *Unit 2*). If $g: (-1, 1) \to \mathbf{R}$ is defined by

$$g(y) = \frac{y}{1 - |y|}, \quad y \in (-1, 1),$$

then g is continuous and it is easily checked that $f(g(y)) = y$ if $y \in (0, 1)$ and $g(f(x)) = x$ if $x \in \mathbf{R}$. Thus f is a continuous bijection and $f^{-1} = g$ is continuous. Hence f is a topological equivalence. ■

3.7 Objectives of Unit 3

After working through this unit, you should know:

(1) the definitions of a metric on a set and a metric space, and be able to show whether or not a given function $d: A \times A \to \mathbf{R}$ is a metric on the set A;

(2) the definition of continuity of a function between metric spaces, and be able to apply it to examples;

(3) the definitions of a metric subspace and of the product of metric spaces;

(4) the definitions of a bounded set in a metric space and of its diameter;

(5) the definitions of open balls and of open sets in a metric space, and the main properties of open sets;

(6) the characterizations of continuity in terms of open balls and open sets;

(7) the definitions and significance of topological and Lipschitz equivalence of metrics and the relation between them, and be able to determine whether or not given metrics are topologically equivalent;

(8) the definitions of a topological equivalence (homeomorphism), a Lipschitz equivalence and an isometry between metric spaces, and the relations between them.

3.8 Revision Questions

SAQ 24 Prove that if $d\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined by

$$d(x, y) = \begin{cases} |x - y| + 1, & \text{if exactly one of } x \text{ and } y \text{ is strictly positive,} \\ |x - y|, & \text{otherwise,} \end{cases}$$

then $\{\mathbf{R}, d\}$ is a metric space. Find the open balls $B_1(0)$, $B_2(0)$ and $B_3(1)$.

SAQ 25 Let M be a metric space and let $f, g\colon M \to \mathbf{R}$ be continuous, where $\mathbf{R}$ has the Euclidean metric. Prove that:

(i) $|f|$ is continuous;

(ii) if $s, t \in \mathbf{R}$ then $sf + tg$ is continuous;

(iii) Max $\{f, g\}$ and Min $\{f, g\}$ are continuous.

SAQ 26 (a) Prove that the rectangle $(a, b) \times (c, d)$ is open in $\mathbf{R}^2$ (with the Euclidean metric).

(b) Let V and W be open in the metric spaces $\{A, d\}$ and $\{A', d'\}$ respectively. Prove that $V \times W$ is open in the metric space $\{A \times A', d_\infty\}$ where d_∞ is the metric on $A \times A'$ defined in **S** Example 2.2.7.

SAQ 27 Let A be the subspace of the metric space $\{\mathscr{C}[0, 1], d_1\}$ consisting of the continuous functions

$$f_n\colon x \mapsto x^{n-1}, \quad x \in [0, 1],$$

for each positive integer n. Prove that A is bounded.

SAQ 28 Let V be open in $\mathbf{R}$. Prove that the set

$$\{f \in \mathscr{C}[a, b]\colon f(a) \in V\}$$

is open in $\{\mathscr{C}[a, b], d_\infty\}$. [Hint: Recall the result of SAQ 13 and use **S** Proposition 2.3.13.]

SAQ 29 The function $e\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is defined as follows: if $x = y$ then $e(x, y) = 0$, whilst if $x \neq y$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ then $e(x, y) = \dfrac{1}{k}$ if $x_i = y_i$ for $i < k$ and $x_k \neq y_k$. Show that e is a metric on $\mathbf{R}^n$ which is Lipschitz equivalent to the discrete metric on $\mathbf{R}^n$.

SAQ 30 A *rotation* of $\mathbf{R}^2$ is a map $\rho\colon \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$\rho(x) = (x_1 \cos\theta - x_2 \sin\theta,\ x_1 \sin\theta + x_2 \cos\theta), \quad x = (x_1, x_2) \in \mathbf{R}^2,$$

where θ is a fixed real number. Prove that ρ is an isometry of the Euclidean metric space $\mathbf{R}^2$ with itself.

SOLUTION TO SAQ 24 Since $d(x, y) \geq |x - y|$ it follows that $d(x, y) \geq 0$ for all x, y in $\mathbf{R}$ and $d(x, y) > 0$ if $x \neq y$. Also $d(x, x) = 0$ for all x in $\mathbf{R}$. Thus (M1) is satisfied. Plainly $d(x, y) = d(y, x)$ for all x, y in $\mathbf{R}$ so that (M2) is satisfied. To check that (M3) is satisfied, let x, y and z be distinct real numbers. If it is not the case that precisely one of x and z is strictly positive, then $d(x, z) = |x - z|$. But $d(x, y) \geq |x - y|$ and $d(y, z) \geq |y - z|$.

Since

$$|x - z| \leq |x - y| + |y - z|$$

it follows that

$$d(x, z) \leq d(x, y) + d(y, z).$$

To complete the verification of (M3), it is enough to consider the case in which $x \le 0$ and $z > 0$. Then $d(x,z) = |x - z| + 1$. If $y \le 0$ then $d(x,y) = |x - y|$ and $d(y,z) = |y - z| + 1$, whilst if $y > 0$ then $d(x,y) = |x - y| + 1$ and $d(y,z) = |y - z|$; so that in both cases we have

$$d(x,z) \le d(x,y) + d(y,z).$$

Thus d is a metric on **R**. We have

$$\begin{aligned} B_1(0) &= \{x \in \mathbf{R}: -1 \le x < 0\} = (-1,0], \\ B_2(0) &= \{x \in \mathbf{R}: -2 < x < 1\} = (-2,1), \text{ and} \\ B_3(1) &= \{x \in \mathbf{R}: -1 < x < 4\} = (-1,4). \quad \blacksquare \end{aligned}$$

SOLUTION TO SAQ 25 Let $M = \{A, d\}$ and let $a \in A$.

(i) Let $\varepsilon > 0$ be given. Since f is continuous at a there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for every x in A such that $d(x,a) < \delta$. But

$$\big||f(x)| - |f(a)|\big| \le |f(x) - f(a)|$$

by **S** Corollary 1.1.10. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\big||f|(x) - |f|(a)\big| < \varepsilon$ for every x in A such that $d(x,a) < \delta$. Thus $|f|$ is continuous at a. Since this is true for every a in A, we see that $|f|: M \to \mathbf{R}$ is continuous.

(ii) First we show that $f + g$ is continuous at a. Let $\varepsilon > 0$ be given. Since $(f + g)(x) = f(x) + g(x), x \in A$, we see that

$$\begin{aligned} |(f+g)(x) - (f+g)(a)| &= |(f(x) - f(a)) + (g(x) - g(a))| \\ &\le |f(x) - f(a)| + |g(x) - g(a)|. \end{aligned}$$

There exists $\delta_1 > 0$ such that $|f(x) - f(a)| < \dfrac{\varepsilon}{2}$ for every x in A such that $d(x,a) < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - g(a)| < \dfrac{\varepsilon}{2}$ for every x in A such that $d(x,a) < \delta_2$. Let $\delta = \text{Min}\,\{\delta_1, \delta_2\}$. Then $\delta > 0$ and if $x \in A$ and $d(x,a) < \delta$ then

$$|(f+g)(x) - (f+g)(a)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $f + g$ is continuous at a. Next we show that sf is continuous at a. If $s = 0$ then sf is the constant function $x \mapsto 0$, $x \in A$, and so is continuous at a. Thus we suppose that $s \ne 0$. Let $\varepsilon > 0$ be given. Now

$$|(sf)(x) - (sf)(a)| = |sf(x) - sf(a)| = |s|\,|f(x) - f(a)|.$$

There exists $\delta > 0$ such that $|f(x) - f(a)| < \dfrac{\varepsilon}{|s|}$ for every x in A such that $d(x,a) < \delta$. Thus if $x \in A$ and $d(x,a) < \delta$ then $|(sf)(x) - (sf)(a)| < \varepsilon$ so that sf is continuous at a. It now follows that $sf + tg$ is continuous at a.

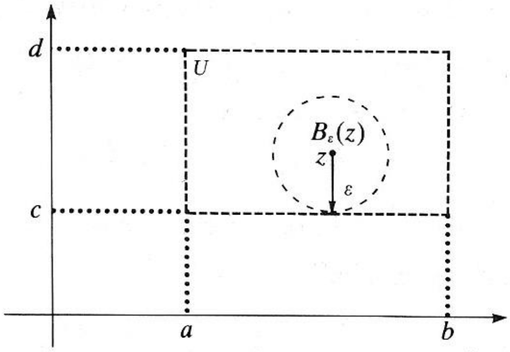
Since this is true for each a in A we see that $sf + tg: M \to \mathbf{R}$ is continuous.

(iii) This is an immediate consequence of (i) and (ii), since

$\text{Max}\,\{f, g\} = \frac{1}{2}(f + g + |f - g|)$, $\text{Min}\,\{f, g\} = \frac{1}{2}(f + g - |f - g|)$. $\blacksquare$

SOLUTION TO SAQ 26

(a)



Let $z = (z_1, z_2) \in U = (a,b) \times (c,d)$. Then $a < z_1 < b$ and $c < z_2 < d$. Now let

$$\varepsilon = \text{Min}\,\{z_1 - a,\ b - z_1,\ z_2 - c,\ d - z_2\}.$$

Then $B_\varepsilon(z) \subset U$.

(b) Let $a = (a_1, a_2) \in V \times W$. Then $a_1 \in V$ which is open in $\{A, d\}$ so there exists $\varepsilon_1 > 0$ such that $d(x_1, a_1) < \varepsilon_1$ implies that $x_1 \in V$. Also $a_2 \in W$ so that there exists $\varepsilon_2 > 0$ such that $d'(x_2, a_2) < \varepsilon_2$ implies that $x_2 \in W$.

If $x = (x_1, x_2) \in A \times A'$ then

$$d_\infty(x, a) = \text{Max}\,\{d(x_1, a_1), d'(x_2, a_2)\}.$$

Now let $\varepsilon = \text{Min}\{\varepsilon_1, \varepsilon_2\}$. If $x = (x_1, x_2) \in A \times A'$ and $d_\infty(x, a) < \varepsilon$ then

$$d(x_1, a_1) \le d_\infty(x, a) < \varepsilon \le \varepsilon_1$$

so that $x_1 \in V$, and similarly

$$d'(x_2, a_2) \le d_\infty(x, a) < \varepsilon \le \varepsilon_2$$

so that $x_2 \in W$. It follows that in the metric space $\{A \times A', d_\infty\}$ we have $B_\varepsilon(a) \subset V \times W$. Thus for each a in $V \times W$ there exists some $\varepsilon > 0$ such that $B_\varepsilon(a) \subset V \times W$. Hence $V \times W$ is open. ■

SOLUTION TO SAQ 27

If $n \in \mathbf{N}$ then

$$d_1(f_1, f_n) = \int_0^1 |f_1(x) - f_n(x)|dx = \int_0^1 |1 - x^{n-1}|dx$$

$$= \int_0^1 (1 - x^{n-1})dx = 1 - \frac{1}{n} \le 1. \quad ■$$

SOLUTION TO SAQ 28

It was shown in SAQ 13 that $\theta\colon \mathscr{C}[a,b] \to \mathbf{R}$, given by $\theta(f) = f(a)$, $f \in \mathscr{C}[a,b]$, is continuous with respect to the metric d_∞ on $\mathscr{C}[a,b]$ and the Euclidean metric on $\mathbf{R}$. It follows from **S** Proposition 2.3.13 that $\theta^{-1}(V)$ is open in $\{\mathscr{C}[a,b], d_\infty\}$. But

$$\theta^{-1}(V) = \{f \in \mathscr{C}[a,b] : \theta(f) \in V\} = \{f \in \mathscr{C}[a,b] : f(a) \in V\}. \quad ■$$

SOLUTION TO SAQ 29

It is clear that (M1) and (M2) hold. Now let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ be distinct elements of $\mathbf{R}^n$. If $e(x,z) = \dfrac{1}{k}$ then $x_k \ne z_k$ so that either $y_k \ne x_k$ or $y_k \ne z_k$ from which it follows that either $e(x,y) \ge \dfrac{1}{k}$ or $e(y,z) \ge \dfrac{1}{k}$.

Therefore

$$e(x,z) \le e(x,y) + e(y,z).$$

Thus e is a metric on $\mathbf{R}^n$.

Let d denote the discrete metric on $\mathbf{R}^n$. If $x = y$ then $e(x, y) = d(x, y) = 0$. If $x \neq y$ then $d(x, y) = 1$ whilst $\frac{1}{n} \leq e(x, y) \leq 1$. Therefore

$$\frac{1}{n} d(x, y) \leq e(x, y) \leq d(x, y)$$

for all x, y in $\mathbf{R}^n$. Thus d and e are Lipschitz equivalent. ■

SOLUTION TO SAQ 30 Since ρ is a bijection, it remains to show that

$$d_2(\rho(x), \rho(y)) = d_2(x, y)$$

for all x, y in $\mathbf{R}^2$.

If $x = (x_1, x_2) \in \mathbf{R}^2$, let us put

$$\rho_1(x) = x_1 \cos\theta - x_2 \sin\theta, \quad \rho_2(x) = x_1 \sin\theta + x_2 \cos\theta$$

so that $\rho(x) = (\rho_1(x), \rho_2(x))$. Now let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points of $\mathbf{R}^2$. Then

$$d_2(\rho(x), \rho(y)) = \{(\rho_1(x) - \rho_1(y))^2 + (\rho_2(x) - \rho_2(y))^2\}^{\frac{1}{2}}.$$

But

$$\begin{aligned}(\rho_1(x) - \rho_1(y))^2 &= ((x_1 - y_1)\cos\theta - (x_2 - y_2)\sin\theta)^2 \\ &= t_1^2 \cos^2\theta - 2t_1t_2 \sin\theta\cos\theta + t_2^2 \sin^2\theta\end{aligned}$$

where $t_1 = x_1 - y_1$ and $t_2 = x_2 - y_2$. Similarly,

$$(\rho_2(x) - \rho_2(y))^2 = t_1^2 \sin^2\theta + 2t_1t_2 \sin\theta\cos\theta + t_2^2 \cos^2\theta.$$

Thus

$$\begin{aligned}d_2(\rho(x), \rho(y)) &= \{t_1^2(\cos^2\theta + \sin^2\theta) + t_2^2(\cos^2\theta + \sin^2\theta)\}^{\frac{1}{2}} \\ &= \{t_1^2 + t_2^2\}^{\frac{1}{2}} = \{(x_1 - y_1)^2 + (x_2 - y_2)^2\}^{\frac{1}{2}} \\ &= d_2(x, y). \quad \blacksquare\end{aligned}$$

UNIT 4
TOPOLOGICAL SPACES I

4.0 Introduction

In *Unit 3* Section 3.5, certain subsets of a metric space $\{A, d\}$ called d-open sets were identified. Two important properties of d-open sets were established in S Proposition 2.3.17 and S Proposition 2.3.19.

In this unit, the process of abstraction is taken a step further. A *topological space*, $\{A, \mathscr{T}\}$, consists of a set A and a collection $\mathscr{T}$ of subsets of A, called a *topology* for A. The topology $\mathscr{T}$ will be required, by definition, to satisfy certain conditions motivated by the properties of d-open sets in a metric space. If $\{A, d\}$ is a metric space, then, as we shall see, the collection of d-open sets is a topology for A. In Section 4.1, the definition of a topological space is analysed and some examples are studied. It turns out that, whilst every metric space is a topological space, the class of topological spaces is much larger than the class of metric spaces.

In S Proposition 2.3.13, continuity of a function between metric spaces was characterized in terms of open sets of the domain and codomain, i.e. in terms of the topologies of the domain and codomain. This result motivates our *definition* of continuity of a function between topological spaces. *Continuous functions* between topological spaces are studied in Section 4.2.

In *Unit 3* Section 3.2, we saw that each subset of a metric space M can be given the structure of a metric space, the resulting objects being known as subspaces of M. Likewise, each subset of a topological space T can be given the structure of a topological space, and the subset together with its structure will be called a *subspace* of T. Subspaces are studied in Section 4.4.

Given topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ for sets A_1 and A_2 respectively, there is a related topology for the Cartesian product $A_1 \times A_2$. Thus the notion of a *product* of topological spaces can be introduced. Topological products are studied in Section 4.5.

4.1 Topological Spaces

In this section, two very important ideas are introduced: that of a topological space in S Definition 3.1.1, and that of a continuous map between topological spaces in S Definition 3.1.2. We shall concentrate on topological spaces and turn to the study of continuity in the next section.

READ S from the beginning of Section 3.1, page 45 to page 48, line 5.

NOTES (i) Strong motivation for the definition of a topology comes from the properties of d-open sets in a metric space $\{A, d\}$. In that context, 'openness' has geometrical significance. We *stress*, however, that a topology $\mathscr{T}$ for a set A is simply a collection $\mathscr{T}$ of subsets of A which satisfies the conditions (T1), (T2) and (T3). In SAQ 16 of *Unit 3* you saw that open balls in that metric space were rather different from what you might have expected using 'native geometric wit'. In topological spaces too, open sets have generally no particular geometric interpretation.

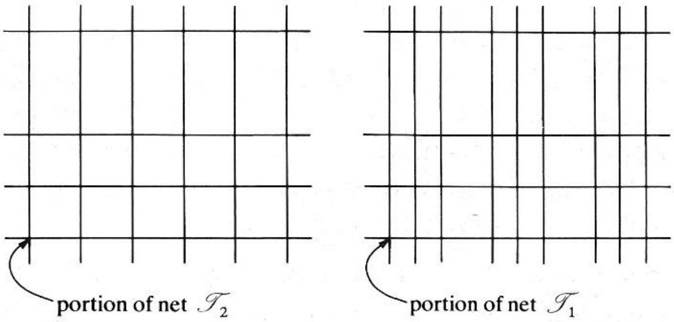
It is important to note that in a topological space, while the union of any number of open sets is again an open set, the axioms guarantee only that the *intersection* of a *finite* number of open sets is an open set.

(ii) *Page 46, line −1*
It is a consequence of S Proposition 2.4.2 that metric spaces $M_1 = \{A, d_1\}$ and $M_2 = \{A, d_2\}$, with the same set of points A, give rise to the same topological space $\{A, \mathscr{T}\}$ if and only if the metrics d_1 and d_2 on A are

topologically equivalent. The metrics d_1, d_2 and d_∞ on $\mathbf{R}^n$ are Lipschitz equivalent, hence topologically equivalent, and so give rise to the same topology on $\mathbf{R}^n$ (which we shall refer to as the *usual topology*).

(iii) *Page 47, lines 6–14*

It will be useful to us later if we augment the terminology of **S** here. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies on the same set. If $\mathscr{T}_1 \supset \mathscr{T}_2$ and $\mathscr{T}_1 \neq \mathscr{T}_2$, we shall say that $\mathscr{T}_1$ is *strictly finer* than $\mathscr{T}_2$ or, equivalently, that $\mathscr{T}_2$ is *strictly coarser* than $\mathscr{T}_1$. In other words, $\mathscr{T}_1$ is strictly finer than $\mathscr{T}_2$ if $\mathscr{T}_1$ contains every set that is in $\mathscr{T}_2$ and some more besides. The terminology arises from fishermen's nets: one net, $\mathscr{T}_1$, has a strictly finer mesh than another, $\mathscr{T}_2$, if it has all the mesh (i.e. intersections) that $\mathscr{T}_2$ has and some more besides.


portion of net $\mathscr{T}_2$
portion of net $\mathscr{T}_1$

Also, if neither $\mathscr{T}_1 \supset \mathscr{T}_2$ nor $\mathscr{T}_2 \supset \mathscr{T}_1$ is true, then we say that the topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ are *non-comparable*. You will be able to find many examples of non-comparable topologies amongst your answers to SAQ 2.

(iv) *Page 47, line −5 to page 48, line 5*

The topological space $(\mathbf{R}, \mathscr{T})$ introduced in **S** Example 3.1.7 is an example of a *non-metrizable space*.

To see this we first note that if U and V are any non-empty members of $\mathscr{T}$ then $U \cap V \neq \varnothing$. For $\mathbf{R} - U$ and $\mathbf{R} - V$ are finite sets and, by De Morgan's Laws (Proposition 3 of *Unit 1*),

$$\mathbf{R} - (U \cap V) = (\mathbf{R} - U) \cup (\mathbf{R} - V),$$

so that $U \cap V$ is an infinite set.

Now if $(\mathbf{R}, \mathscr{T})$ were a metrizable space, $\mathscr{T}$ would be the collection of d-open sets corresponding to some metric d on $\mathbf{R}$. Then, as we saw in SAQ 19 of *Unit 3*, there would exist non-empty members U and V of $\mathscr{T}$ such that $U \cap V = \varnothing$. This contradiction shows that $(\mathbf{R}, \mathscr{T})$ is a non-metrizable space. ▲

Remark. The work necessary to check that a subset $\mathscr{T}$ of 2^A that satisfies (T1) is a topology for the set A can be reduced by the following observations.

(i) To establish that (T2) holds, it is necessary to check only that if $V, W \in \mathscr{T} - \{\varnothing, A\}$ then $V \cap W \in \mathscr{T}$.

For if $V \in \mathscr{T}$ then

$$V \cap \varnothing = \varnothing \quad \text{and} \quad V \cap A = V.$$

(ii) To establish that (T3) holds, it is necessary to check only that if $(V_i)_{i \in I}$ is a family of members of $\mathscr{T} - \{\varnothing, A\}$, then $\bigcup_{i \in I} V_i \in \mathscr{T}$.

For if $(V_i)_{i \in I}$ is a family of members of $\mathscr{T}$ and there is some $k \in I$ such that $V_k = \varnothing$, then, putting $J = I - \{k\}$ we have $\bigcup_{i \in I} V_i = \bigcup_{j \in J} V_j$; and if there exists some $k \in I$ such that $V_k = A$ then $\bigcup_{i \in I} V_i = A$.

The gist of (i) and (ii) is that, once you have checked that $\varnothing$ and A both belong to the collection $\mathcal{T}$, you can forget about them while checking (T2) and (T3).

EXAMPLE 1 Let $A = \{a, b, c\}$ and let

$$\mathcal{T} = \{\varnothing, \{a\}, \{b, c\}, A\}.$$

Then $\{A, \mathcal{T}\}$ is a topological space.

Condition (T1) is plainly satisfied. To see that (T2) and (T3) are satisfied, we need only observe that

$$\{a\} \cap \{b, c\} = \varnothing \in \mathcal{T} \quad \text{and} \quad \{a\} \cup \{b, c\} = A \in \mathcal{T}.$$

We leave you to check that $(A, \mathcal{T}_1)$, where $\mathcal{T}_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, A\}$, is a topological space. Since $\mathcal{T}$ is a *proper* subset of $\mathcal{T}_1$, the topology $\mathcal{T}_1$ on A is strictly finer than the topology $\mathcal{T}$.

EXAMPLE 2 If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set A then $\mathcal{T}_1 \subset 2^A$, and $\mathcal{T}_2 \subset 2^A$ so that $\mathcal{T}_1 \cup \mathcal{T}_2 \subset 2^A$ and $\mathcal{T}_1 \cap \mathcal{T}_2 \subset 2^A$. (Recall that $\mathcal{T}_1 \cup \mathcal{T}_2$ consists of those subsets of A which belong either to $\mathcal{T}_1$ or to $\mathcal{T}_2$, and that $\mathcal{T}_1 \cap \mathcal{T}_2$ of those which belong to both $\mathcal{T}_1$ and $\mathcal{T}_2$.) The following questions arise fairly naturally:

(i) Is $\mathcal{T}_1 \cup \mathcal{T}_2$ necessarily a topology on A?

(ii) Is $\mathcal{T}_1 \cap \mathcal{T}_2$ necessarily a topology on A?

(i) While $\mathcal{T}_1 \cup \mathcal{T}_2$ may be a topology on A, here is an example of topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for which $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology.

Let $A = \{a, b, c\}$, $\mathcal{T}_1 = \{\varnothing, \{a\}, A\}$, and $\mathcal{T}_2 = \{\varnothing, \{b\}, A\}$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on A (recall the above Remark).

But $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, \{a\}, \{b\}, A\}$ is not a topology on A. For $\{a\}$, $\{b\}$ belong to $\mathcal{T}_1 \cup \mathcal{T}_2$, whilst $\{a\} \cup \{b\} = \{a, b\}$ does not belong to $\mathcal{T}_1 \cup \mathcal{T}_2$.

(ii) $\mathcal{T}_1 \cap \mathcal{T}_2$ is always a topology on A. This fact is a special case of the following Proposition, which we prove instead, as its proof is no harder than the proof of (ii) and the actual result will be useful later on.

PROPOSITION Let $(\mathcal{T}_i)_{i \in I}$ be an indexed family of topologies on a set A. Then if

$$\mathcal{T} = \{V \in 2^A : V \in \mathcal{T}_i \text{ for every } i \text{ in } I\},$$

$\mathcal{T}$ is a topology on A.

PROOF By (T1), $\varnothing \in \mathcal{T}_i$ and $A \in \mathcal{T}_i$ for every i in I so that $\varnothing \in \mathcal{T}$ and $A \in \mathcal{T}$. Thus $\mathcal{T}$ satisfies condition (T1).

If $U, V \in \mathcal{T}$, then $U, V \in \mathcal{T}_i$ for every i in I so that $U \cap V \in \mathcal{T}_i$ for every i in I and hence $U \cap V \in \mathcal{T}$. Thus $\mathcal{T}$ satisfies condition (T2).

To verify (T3), let $(V_j)_{j \in J}$ be a family of members of $\mathcal{T}$. If $i \in I$, then $V_j \in \mathcal{T}_i$ for every j in J so that $\bigcup\limits_{j \in J} V_j \in \mathcal{T}_i$, since $\mathcal{T}_i$ satisfies (T3). Since this is true for every i in I, it follows that $\bigcup\limits_{j \in J} V_j \in \mathcal{T}$. Thus $\mathcal{T}$ satisfies condition (T3). Hence $\mathcal{T}$ is a topology on A.

EXAMPLE 3 Let $\mathcal{T}$ be the collection of subsets of $\mathbf{R}$ which consists of all sets V such that either $0 \notin V$ or $\mathbf{R} - V$ is finite. (For example, $\mathbf{N} \in \mathcal{T}$ since $0 \notin \mathbf{N}$; if $V = \mathbf{R} - \{2, 3, 5\}$ then $V \in \mathcal{T}$ since $\mathbf{R} - V = \{2, 3, 5\}$ is finite; if $W = \mathbf{R} - \{0\}$ then $W \in \mathcal{T}$ since $0 \notin W$ (and $\mathbf{R} - W$ is finite!); $\mathbf{Q} \notin \mathcal{T}$ since $0 \in \mathbf{Q}$ and $\mathbf{R} - \mathbf{Q}$ is infinite.)

Then $\mathcal{T}$ is a topology on $\mathbf{R}$.

For, firstly, (T1) is satisfied, as $\varnothing \in \mathcal{T}$ since $0 \notin \varnothing$ and $\mathbf{R} \in \mathcal{T}$ since $\mathbf{R} - \mathbf{R} = \varnothing$ (recall the convention, mentioned in *Unit 1* Section 1.5, that $\varnothing$ is a finite set).

To see that (T2) is satisfied, let V and W be members of $\mathscr{T}$. There are two cases to deal with: (i) 0 belongs to both V and W; (ii) 0 fails to belong to at least one of V, W. In case (i), $\mathbf{R} - V$ and $\mathbf{R} - W$ are both finite. Then by Proposition 3 of *Unit 1*,

$$\mathbf{R} - (V \cap W) = (\mathbf{R} - V) \cup (\mathbf{R} - W),$$

so that $\mathbf{R} - (V \cap W)$ is finite; it follows that, in this case, $V \cap W \in \mathscr{T}$. In case (ii), $0 \notin V \cap W$, so that again $V \cap W \in \mathscr{T}$.

Finally, let $(V_i)_{i \in I}$ be an indexed family of members of $\mathscr{T}$ and let $V = \bigcup_{i \in I} V_i$. If, for each i, $0 \notin V_i$ then $0 \notin V$ so that $V \in \mathscr{T}$. If there exists j in I such that $0 \in V_j$, then $\mathbf{R} - V_j$ is finite. But, by Proposition 3 of *Unit 1*,

$$\mathbf{R} - V = \mathbf{R} - (\bigcup_{i \in I} V_i) = \bigcap_{i \in I} (\mathbf{R} - V_i) \subset \mathbf{R} - V_j,$$

so that $\mathbf{R} - V$ is finite and again $V \in \mathscr{T}$. Hence (T3) is satisfied. Therefore $(\mathbf{R}, \mathscr{T})$ is a topological space.

Notice that the topology $\mathscr{T}$ is non-comparable with the usual topology for $\mathbf{R}$. For let $G = [1, \infty)$ and $H = (-1, 1)$. Then G is not a member of the usual topology for $\mathbf{R}$ but $G \in \mathscr{T}$ since $0 \notin G$. Also, H is a member of the usual topology for $\mathbf{R}$ but $H \notin \mathscr{T}$ since $0 \in H$ and $\mathbf{R} - H$ is infinite.

Now play the first part of the tape for Unit 4

SAQ 1 Let $A = \{a, b, c, d, e\}$. Determine whether or not each of the following sets of subsets of A is a topology on A.

(i) $\mathscr{T}_1 = \{\varnothing, \{a\}, \{a,b\}, \{a,c\}, A\}$

(ii) $\mathscr{T}_2 = \{\varnothing, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}, A\}$

(iii) $\mathscr{T}_3 = \{\varnothing, \{a\}, \{a,b\}, \{a;b,c\}, \{a,b,c,d\}\}$

SAQ 2 Find all possible topologies on

(i) a set which has two elements a, b;

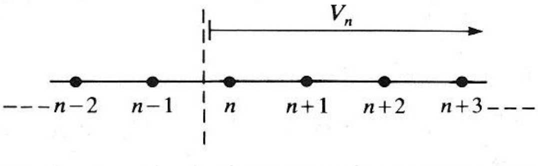
(ii) a set which has three elements a, b, c.

(Do not spend more than 10 minutes on this question before looking at the solution.)

SAQ 3 Let $\mathscr{T}$ be the set of subsets of $\mathbf{N}$ consisting of $\varnothing$ and all subsets of $\mathbf{N}$ of the form

$$V_n = \{k \in \mathbf{N} \colon k \geq n\}, \quad n \in \mathbf{N}.$$

Show that $\mathscr{T}$ is a topology on $\mathbf{N}$. Is the topological space $\{\mathbf{N}, \mathscr{T}\}$ metrizable?



SAQ 4 Show that a subset V of a topological space T is an open set of T if and only if for each x in V there exists an open set V_x of T such that $x \in V_x \subset V$.

SOLUTION TO SAQ 1

(i) $\mathscr{T}_1$ is not a topology on A. For example, $\{a,b\} \in \mathscr{T}_1$, $\{a,c\} \in \mathscr{T}_1$ but $\{a,b\} \cup \{a,c\} = \{a,b,c\} \notin \mathscr{T}_1$.

(ii) $\mathscr{T}_2$ is not a topology on A since $\{a, b, c\} \in \mathscr{T}_2$, $\{a, b, d\} \in \mathscr{T}_2$ but $\{a,b,c\} \cap \{a,b,d\} = \{a,b\} \notin \mathscr{T}_2$.

(iii) $\mathscr{T}_3$ is not a topology on A since $A \notin \mathscr{T}_3$. ■

OLUTION TO SAQ 2

(i) Let $A = \{a, b\}$. Then we can list the sets in 2^A as follows:

$$\varnothing, \{a\}, \{b\}, A.$$

Using the Remark after Note (iv), it is easy to check that there are exactly 4 topologies on A; namely, the indiscrete topology, the discrete topology, and the topologies

$$\{\varnothing, \{a\}, A\} \quad \text{and} \quad \{\varnothing, \{b\}, A\}.$$

For example $\{\{a\}, \{a,b\}, A\}$ is a subset of 2^A but is *not* a topology since (T1) does not hold for it.

(ii) Let $A = \{a, b, c\}$. There are 29 topologies on A. To ease the task of listing them, let us say that two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on A are of the same *type* if some permutation of the symbols a, b, c transforms one to the other. For example, the topologies

$$\{\varnothing, \{a\}, \{b,c\}, A\} \quad \text{and} \quad \{\varnothing, \{b\}, \{a,c\}, A\}$$

are of the same type. Now we can list the 29 topologies on A:

Indiscrete topology;
3 topologies of type $\{\varnothing, \{a\}, A\}$;
3 topologies of type $\{\varnothing, \{a,b\}, A\}$;
6 topologies of type $\{\varnothing, \{a\}, \{a,b\}, A\}$;
3 topologies of type $\{\varnothing, \{a\}, \{b,c\}, A\}$;
3 topologies of type $\{\varnothing, \{a\}, \{b\}, \{a,b\}, A\}$;
3 topologies of type $\{\varnothing, \{a\}, \{a,b\}, \{a,c\}, A\}$;
6 topologies of type $\{\varnothing, \{a\}, \{b\}, \{a,b\}, \{a,c\}, A\}$;
Discrete topology. ■

SOLUTION TO SAQ 3

Since $\varnothing \in \mathscr{T}$, by definition, and $\mathbf{N} = V_1 \in \mathscr{T}$, we see that condition (T1) is satisfied.

If n, $m \in \mathbf{N}$ and $n < m$ then $V_m \subset V_n$ so that

$$V_n \cap V_m = V_m.$$

Hence (T2) is satisfied.

If I is a non-empty subset of $\mathbf{N}$, then I has a least member, i.e. there exists k in I such that $k \leq n$ for each n in I. Then $V_n \subset V_k$ for each n in I so that

$$\bigcup_{n \in I} V_n = V_k.$$

Hence (T3) is satisfied.

Therefore $(\mathbf{N}, \mathscr{T})$ is a topological space. It is not metrizable since no two non-empty open sets are disjoint (from which we obtain a contradiction just as in Note (iv)). ■

SOLUTION TO SAQ 4

If V is an open set of T then we can take $V_x = V$ for each x in V. Conversely, suppose that V is a subset of T with the property that for each x in V there exists an open set V_x such that $x \in V_x \subset V$. Then (by SAQ 10 of *Unit 1*)

$$\bigcup_{x \in V} V_x = V.$$

Since V is a union of open sets of T, it follows that V is an open set of T.

4.2 Continuity

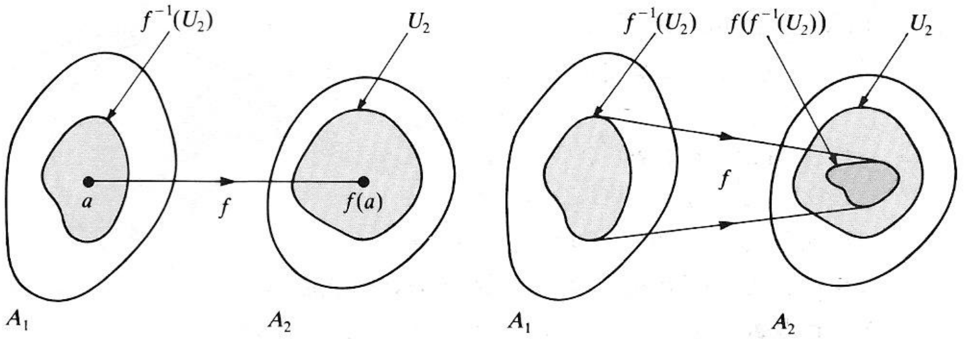
READ S from the beginning of Section 3.1 on page 45 to page 48, line -9.

NOTES (i) *Page 45, line 1*

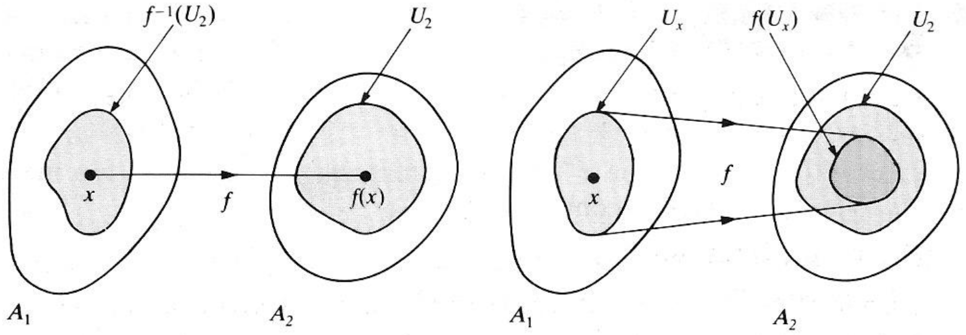
We have asked you to re-read most of Section 3.1 because of the importance of the ideas introduced there. In this section we are going to study continuity of functions whose domain and codomain are topological spaces.

(ii) *Page 46, line 9*

Let us show that a function is continuous if and only if it is continuous at each point of its domain. Let $(A_1, \mathscr{T}_1)$ and $(A_2, \mathscr{T}_2)$ be topological spaces and let $f: A_1 \to A_2$ be a function.



Suppose first that f is continuous. If $a \in A_1$, $U_2 \in \mathscr{T}_2$ and $f(a) \in U_2$ then $a \in f^{-1}(U_2)$ and, since f is continuous, $f^{-1}(U_2) \in \mathscr{T}_1$. But $f(f^{-1}(U_2)) \subset U_2$ (by SAQ 14(d) of *Unit 1*) so that f is continuous at a. Thus, if f is continuous, then f is continuous at every point of A_1.



Conversely, suppose that f is continuous at every point of A_1 and let $U_2 \in \mathscr{T}_2$. We must show that $f^{-1}(U_2) \in \mathscr{T}_1$. If $x \in f^{-1}(U_2)$ then $f(x) \in U_2$ so that, since f is continuous at x, there exists U_x in $\mathscr{T}_1$ such that $x \in U_x$ and $f(U_x) \subset U_2$. Since $f(U_x) \subset U_2$ it follows that $U_x \subset f^{-1}(U_2)$. Hence, for each x in $f^{-1}(U_2)$ there exists U_x in $\mathscr{T}_1$ such that $x \in U_x \subset f^{-1}(U_2)$. It follows (by SAQ 4) that $f^{-1}(U_2) \in \mathscr{T}_1$. Therefore, if f is continuous at each point of A_1 then f is continuous.

As you will see in the remainder of the course, continuity at a point is not a very natural topological notion, whereas Definition 3.1.2 is.

(iii) *Page 48, Proposition 3.1.8*

Let $T_1 = \{A_1, \mathscr{T}_1\}$ and $T_2 = \{A_2, \mathscr{T}_2\}$ be topological spaces. The statement that $f: T_1 \to T_2$ is continuous is an alternative way of saying that $f: A_1 \to A_2$ is $(\mathscr{T}_1, \mathscr{T}_2)$-continuous. This terminology is employed in the statement of S Proposition 3.1.8.

(iv) *Page 48, line 16*

The result that $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is Proposition 5 of *Unit 1*, Section 1.3.

(v) *Page 48, Proposition 3.1.10(a)*

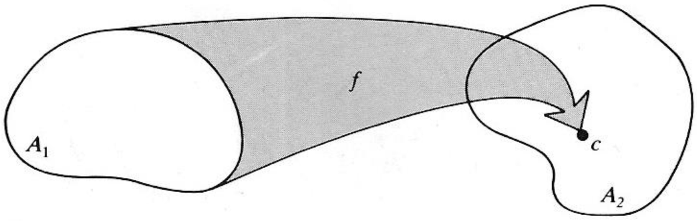
Let A be a set and let $\mathscr{T}_1$ and $\mathscr{T}_2$ be topologies for A. The identity function $id_A : A \to A$ is $(\mathscr{T}_1, \mathscr{T}_2)$-continuous if and only if $id_A^{-1}(V) \in \mathscr{T}_1$ for each V in $\mathscr{T}_2$. But $id_A^{-1}(V) = V$. Thus id_A is $(\mathscr{T}_1, \mathscr{T}_2)$-continuous if and only if $V \in \mathscr{T}_2$ implies that $V \in \mathscr{T}_1$, i.e. if and only if $\mathscr{T}_2 \subset \mathscr{T}_1$. In particular, if $\mathscr{T}$ is any topology for A then id_A is $(\mathscr{T}, \mathscr{T})$-continuous. Putting $T = \{A, \mathscr{T}\}$, we can say that the identity map of T to itself is continuous.

(vi) *Page 48, Proposition 3.1.10(b)*

Let $T_1 = \{A_1, \mathscr{T}_1\}$ and $T_2 = \{A_2, \mathscr{T}_2\}$ be topological spaces, let $c \in A_2$ and let $f : A_1 \to A_2$ be given by

$$f(x) = c, \quad x \in A_1.$$

(This is what we call a *constant map.*)



We must show that $f : T_1 \to T_2$ is continuous.

If $V \in \mathscr{T}_2$ then

$$f^{-1}(V) = \begin{cases} A_1, & \text{if } c \in V \\ \varnothing, & \text{if } c \notin V. \end{cases}$$

Since $A_1 \in \mathscr{T}_1$ and $\varnothing \in \mathscr{T}_1$, it follows that $f^{-1}(V) \in \mathscr{T}_1$ for each V in $\mathscr{T}_2$. Thus $f : T_1 \to T_2$ is continuous. ▲

EXAMPLE 4 Let $T_1 = \{A_1, \mathscr{T}_1\}$ and $T_2 = \{A_2, \mathscr{T}_2\}$ be topological spaces and let $f : A_1 \to A_2$ be a function. Then f is $(\mathscr{T}_1, \mathscr{T}_2)$-continuous if either (i) $\mathscr{T}_1$ is the discrete topology on A_1 (for any $\mathscr{T}_2$) or (ii) $\mathscr{T}_2$ is the indiscrete topology on A_2 (for any $\mathscr{T}_1$).

In case (i), if $V \in \mathscr{T}_2$ then $f^{-1}(V) \in \mathscr{T}_1$ since every subset of A_1 is a member of $\mathscr{T}_1$. Thus f is $(\mathscr{T}_1, \mathscr{T}_2)$-continuous.

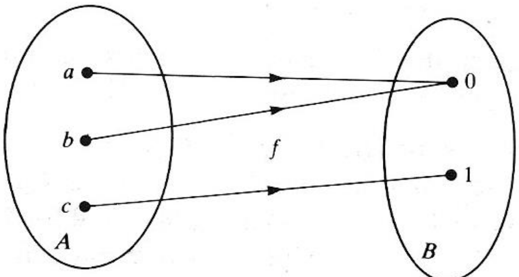
In case (ii), $\mathscr{T}_2 = \{\varnothing, A_2\}$. Since $f^{-1}(A_2) = A_1 \in \mathscr{T}_1$ and $f^{-1}(\varnothing) = \varnothing \in \mathscr{T}_1$, it follows that again f is $(\mathscr{T}_1, \mathscr{T}_2)$-continuous.

EXAMPLE 5 Let $A = \{a, b, c\}$ and let

$$\mathscr{T} = \{\varnothing, \{a\}, \{b, c\}, A\}.$$

As we saw in Example 1, $\{A, \mathscr{T}\}$ is a topological space. Let $B = \{0, 1\}$ and let B have the topology $\mathscr{U} = \{\varnothing, \{0\}, B\}$. Then $f : A \to B$, given by $f(a) = f(b) = 0$, $f(c) = 1$, is not $(\mathscr{T}, \mathscr{U})$-continuous, because $\{0\}\ \mathscr{U}$, but

$$f^{-1}(\{0\}) = \{a, b\}.$$



Since $\{a, b\} \notin \mathscr{T}$, it follows that f is not continuous.

EXAMPLE 6 Let $T_1 = \{A, \mathscr{T}_1\}$ where $A = \{a, b, c\}$ and

$$\mathscr{T}_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, A\}.$$

In Example 1 we saw that $\mathscr{T}_1$ is a topology for A. Let $T_2 = \{\mathbf{R}, \mathscr{T}_2\}$ where $\mathscr{T}_2$ is the topology for $\mathbf{R}$ introduced in Example 3. ($\mathscr{T}_2$ consists of all subsets V of $\mathbf{R}$ such that either $0 \notin V$ or $\mathbf{R} - V$ is finite.) Let us determine which functions $f: T_1 \to T_2$ are continuous, or, equivalently, which functions $f: A \to \mathbf{R}$ are $(\mathscr{T}_1, \mathscr{T}_2)$-continuous.

We observe first that every open set of T_1 which contains c also contains b. (There are only two such sets; notice the cunning use of the word *every*!) Now suppose that $f: A \to \mathbf{R}$ is a function such that $f(b) \neq f(c)$. If $H = \mathbf{R} - \{f(b)\}$ then H is an open set of T_2. But $c \in f^{-1}(H)$ and $b \notin f^{-1}(H)$, so that $f^{-1}(H)$ is not an open set of T_1. Thus, if $f(b) \neq f(c)$ then $f: T_1 \to T_2$ is not continuous; hence if $f: T_1 \to T_2$ is continuous, we must have $f(b) = f(c)$.

In fact, the converse of this is also true. To see this, suppose that $f: A \to \mathbf{R}$ is a function such that $f(b) = f(c) = t \in \mathbf{R}$. If V is an open set of T_2, then there are exactly four possibilities:

$$f^{-1}(V) = \begin{cases} A, & \text{if } t \in V, f(a) \in V, \\ \{b, c\}, & \text{if } t \in V, f(a) \notin V, \\ \{a\}, & \text{if } t \notin V, f(a) \in V, \\ \varnothing, & \text{if } t \notin V, f(a) \notin V. \end{cases}$$

Thus we see that $f^{-1}(V) \in \mathscr{T}_1$ for such V in $\mathscr{T}_2$, and so $f: T_1 \to T_2$ is continuous.

Now play the second part of the tape for Unit 4

SAQ 5 Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be non-comparable topologies on a set A. Show that the identity map id_A on A is neither $(\mathscr{T}_1, \mathscr{T}_2)$-continuous nor $(\mathscr{T}_2, \mathscr{T}_1)$-continuous.

SAQ 6 Let $c \in \mathbf{R}$ and let $\mathscr{T}_c \subset 2^{\mathbf{R}}$ consist of $\varnothing$ together with all subsets V of $\mathbf{R}$ such that $c \in V$. Prove that $\{\mathbf{R}, \mathscr{T}_c\}$ is a topological space. Determine whether or not the function $f: \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = 2x, \quad x \in \mathbf{R},$$

is $(\mathscr{T}_c, \mathscr{T}_c)$-continuous, (i) if $c = 2$, (ii) if $c = 0$.

SAQ 7 Let $\mathscr{T}$ denote the topology for $\mathbf{N}$ introduced in SAQ 3. Prove that, if $f: \mathbf{N} \to \mathbf{N}$ is monotonic increasing, then f is $(\mathscr{T}, \mathscr{T})$-continuous.

SAQ 8 Let $T_1 = \{A, \mathscr{T}_1\}$ be a topological space, and suppose that, for some point $a \in A$, $\{a\}$ is an open set of T_1. Show that f is continuous at a for any topological space T_2 and any function $f: T_1 \to T_2$.

***SAQ 9** Let T be the topological space introduced in **S** Example 3.1.7. Prove that if $f: T \to \mathbf{R}$ is continuous, where $\mathbf{R}$ has the usual topology, then f is a constant map. [Hint: Use a contradiction argument and SAQ 19 of *Unit* 3.]

SOLUTION TO SAQ 5 If $G \subset A$ then

$$id_A^{-1}(G) = G.$$

Since $\mathscr{T}_1$ and $\mathscr{T}_2$ are non-comparable topologies on A, there exists $G \in \mathscr{T}_2$ such that $G \notin \mathscr{T}_1$ and there exists $H \in \mathscr{T}_1$ such that $H \notin \mathscr{T}_2$. Since $G \in \mathscr{T}_2$ and $id_A^{-1}(G) \notin \mathscr{T}_1$, we see that id_A is not $(\mathscr{T}_1, \mathscr{T}_2)$-continuous.
Likewise, since $H \in \mathscr{T}_1$ and $id_A^{-1}(H) \notin \mathscr{T}_2$, we see that id_A is not $(\mathscr{T}_2, \mathscr{T}_1)$-continuous. ■

SOLUTION TO SAQ 6

(T1) is satisfied since $\varnothing \in \mathscr{T}_c$, by definition, and it is clear that $\mathbf{R} \in \mathscr{T}_c$. The intersection of any two subsets of $\mathbf{R}$ containing c contains c and the union of any collection of sets containing c contains c. Thus (T2) and (T3) are also satisfied, so that $\mathscr{T}_c$ is a topology for $\mathbf{R}$.

(i) For any $V \subset \mathbf{R}$,

$$f^{-1}(V) = \{x \in \mathbf{R} : 2x \in V\}.$$

Since $\{2\} \in \mathscr{T}_2$ and $f^{-1}(\{2\}) = \{1\}$, f is not $(\mathscr{T}_2, \mathscr{T}_2)$-continuous as $2 \notin \{1\}$.

(ii) Clearly $f^{-1}(\varnothing) = \varnothing \in \mathscr{T}_0$. Now suppose that $V \in \mathscr{T}_0$ and $V \neq \varnothing$. Then $0 \in V$. But $f(0) = 0$, so that $0 \in f^{-1}(V)$; hence for each V in $\mathscr{T}_0$, $f^{-1}(V) \in \mathscr{T}_0$. Thus f is $(\mathscr{T}_0, \mathscr{T}_0)$-continuous. ■

SOLUTION TO SAQ 7

We recall that $\mathscr{T}$ consists of $\varnothing$ together with all subsets of the form $V_i = \{k \in \mathbf{N} : k \geq i\}$, where $i \in \mathbf{N}$. Combining this with the definition of continuity, we see that a function $f: \mathbf{N} \to \mathbf{N}$ is $(\mathscr{T}, \mathscr{T})$-continuous if and only if, for each positive integer i, either $f^{-1}(V_i) = \varnothing$ or there exists a positive integer $n(i)$ such that

$$f^{-1}(V_i) = V_{n(i)}.$$

Suppose that f is monotonic increasing; that is, that $f(m) \leq f(n)$ for all $m, n \in \mathbf{N}$ with $m \leq n$. If $i \in \mathbf{N}$, then

$$f^{-1}(V_i) = \{k \in \mathbf{N} : f(k) \geq i\}.$$

If $f^{-1}(V_i) \neq \varnothing$, let $n(i)$ be the least integer in $f^{-1}(V_i)$. If $k \in \mathbf{N}$ and $k \geq n(i)$ then $f(k) \geq f(n(i)) \geq i$. Thus $f^{-1}(V_i) = V_{n(i)}$. Hence, if f satisfies the stated condition then f is $(\mathscr{T}, \mathscr{T})$-continuous. ■

Remark. In the opposite direction, we can actually prove that if f is not monotonic increasing then f cannot be $(\mathscr{T}, \mathscr{T})$-continuous.

For if f is not monotonic increasing, there must exist a pair m, n in $\mathbf{N}$ such that $m < n$, and $f(m) > f(n)$. Let $j = f(m)$. Then $m \in f^{-1}(V_j)$, $n \notin f^{-1}(V_j)$ and $m < n$. Thus $f^{-1}(V_j) \neq \varnothing$ and there exists no integer $n(j)$ such that $f^{-1}(V_j) = V_{n(j)}$. So f cannot, in this case, be $(\mathscr{T}, \mathscr{T})$-continuous.

SOLUTION TO SAQ 8

(We use Definition 3.1.3 directly.)

Let U be an open set of T_2 such that $f(a) \in U$. Then $\{a\}$ is an open set of T_1, $a \in \{a\}$ and $f(\{a\}) = \{f(a)\} \subset U$. Therefore f is continuous at a. ■

SOLUTION TO SAQ 9

We recall that the underlying set of T is $\mathbf{R}$ and the open sets of T are $\varnothing$ and all subsets U of $\mathbf{R}$ such that $\mathbf{R} - U$ is a finite set.

Let $f: T \to \mathbf{R}$ be a continuous function. We must prove that f is a constant map. Suppose, on the contrary, that there exist t and u in $\mathbf{R}$ such that $f(t) \neq f(u)$. As the images $f(t)$, $f(u)$ are points in a metric space ($\mathbf{R}$, with the usual topology), by the result of SAQ 19 of *Unit 3*, there must exist disjoint open sets V and W of $\mathbf{R}$ such that $f(t) \in V$ and $f(u) \in W$. Since f is continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are open sets of T and

$$f^{-1}(V) \cap f^{-1}(W) = f^{-1}(V \cap W) = f^{-1}(\varnothing) = \varnothing.$$

But $t \in f^{-1}(V)$ and $u \in f^{-1}(W)$ so that $f^{-1}(V)$ and $f^{-1}(W)$ are disjoint non-empty open sets of T. It was established in Note (iv) of Section 4.1 of this unit, however, that any two non-empty open sets of T have a non-empty intersection. This contradiction shows that f is a constant map. ■

4.3 Bases

In this section we study a useful method for constructing topologies.

READ S all of Section 3.2 from page 49.

NOTES (i) *Page 49, line 2*
That each open set in a metric space is a union of open balls was established in SAQ 18 of *Unit 3*.

(ii) *Page 49, Definition 3.2.1*
Notice that if $\mathscr{B}$ is a basis for topology $\mathscr{T}$ on a set A and $\mathscr{T}_1$ is a topology on A such that $\mathscr{B} \subset \mathscr{T}_1$, then $\mathscr{T} \subset \mathscr{T}_1$. For if $V \in \mathscr{T}$ then $V = \bigcup_{i \in I} B_i$, where $B_i \in \mathscr{B}$ for each i. Then $B_i \in \mathscr{T}_1$ for each i, and it follows from (T3) that $V \in \mathscr{T}_1$. Therefore $\mathscr{T}$ is the coarsest topology on A which contains $\mathscr{B}$. This result often facilitates the comparison of topologies.

(iii) *Page 49, Definition 3.2.2*
It must not be assumed that for a collection $\mathscr{B}$ of subsets of a set A to satisfy (B2) it is necessary that $B_1 \cap B_2 \neq \varnothing$ for all B_1, B_2 in $\mathscr{B}$.

If $B_1, B_2 \in \mathscr{B}$ and $B_1 \cap B_2 = \varnothing$, then we regard $B_1 \cap B_2$ as the union of *no* members of $\mathscr{B}$.

Thus, to prove that a collection $\mathscr{B}$ of subsets of A satisfies (B2), it is enough to show that if $B_1, B_2 \in \mathscr{B}$ and $B_1 \cap B_2 \neq \varnothing$, then $B_1 \cap B_2$ is a union of members of $\mathscr{B}$.

An alternative form of (B2) which is often more convenient to use is the following:

(B2′) If $B_1, B_2 \in \mathscr{B}$ then for each x in $B_1 \cap B_2$, there exists B_x in $\mathscr{B}$ such that $x \in B_x \subset B_1 \cap B_2$.

To see that (B2′) is equivalent to (B2), recall the result of SAQ 10 of *Unit 1*.

(iv) *Page 49, Propositions 3.2.3 and 3.2.4*
These results supply the link between the concepts introduced in Definitions 3.2.1 and 3.2.2. First, we note that a basis for a *given* topology $\mathscr{T}$ on a set A is a collection $\mathscr{B}$ of members of $\mathscr{T}$ which satisfies Definition 3.2.1. Thus $\mathscr{B}$ is a collection of open sets of the topological space $T = \{A, \mathscr{T}\}$ such that each open set of T is a union of members of $\mathscr{B}$.

On the other hand, a synthetic basis for a set A is simply a collection $\mathscr{B}$ of subsets of A which satisfies conditions (B1) and (B2). Given a synthetic basis $\mathscr{B}$ for a set A, the collection $\mathscr{T}$ of all unions of members of $\mathscr{B}$ is a topology on A and $\mathscr{B}$ is a basis for $\mathscr{T}$ in the sense of Definition 3.2.1 (by Proposition 3.2.3 and the first part of Proposition 3.2.4). The second part of Proposition 3.2.4 asserts that if $\mathscr{B}$ is a basis for a given topology $\mathscr{T}$, then $\mathscr{B}$ satisfies conditions (B1) and (B2), so that $\mathscr{B}$ is a synthetic basis for the set A, and the topology which arises from $\mathscr{B}$ as in Proposition 3.2.3 is the given topology $\mathscr{T}$.

(v) *Page 49, line −9*
This formula was established in SAQ 27 of *Unit 1*.

(vi) *Page 50, line 10*
This formula was established in SAQ 15 of *Unit 1*.

(vii) *Page 50, Application 3.2.6*
It is this application which is the main justification for our study of bases. In Examples 9 and 11, we shall use this method to define topologies. You

will see that it would be much more troublesome to describe these topological spaces if the idea of a basis were not available. Notice that if $\mathscr{B}$ is a synthetic basis for a set A and we *define* $\mathscr{T}$ to be the topology on A which has $\mathscr{B}$ as basis, then (by Note (ii)) we obtain the coarsest topology on A which contains $\mathscr{B}$. ▲

EXAMPLE 7 If $\mathbf{R}$ has the Euclidean metric then the collection of all open balls coincides with the collection of all bounded open intervals. Thus the collection of all bounded open intervals is a basis for the usual topology on $\mathbf{R}$.

EXAMPLE 8 Let A be a non-empty set and let $\mathscr{B}$ be the subset of 2^A which consists of all singleton subsets $\{x\}$ where $x \in A$. Then $\mathscr{B}$ is a basis for the discrete topology 2^A on A. For if $V \in 2^A$ then

$$V = \bigcup_{x \in V} \{x\}.$$

In fact, a subset $\mathscr{C}$ of 2^A is a basis for the discrete topology on A if and only if $\mathscr{B} \subset \mathscr{C}$. For if $\mathscr{C}$ is a basis for the discrete topology on A, then $\{x\}$ must be a union of members of $\mathscr{C}$ for each x in A, so that $\{x\} \in \mathscr{C}$ for each x in A.

EXAMPLE 9 Let $\mathscr{B}$ be the set of subsets of $\mathbf{R}$ consisting of all 'half-open' intervals of the form $[a, b)$ where $a < b$. We shall show that $\mathscr{B}$ is a synthetic basis for $\mathbf{R}$.

Since $a \in [a, a+1)$ for such a in $\mathbf{R}$, we see that condition (B1) is satisfied. Let $[a_1, b_1)$ and $[a_2, b_2)$ be two members of $\mathscr{B}$ and suppose that $x \in [a_1, b_1) \cap [a_2, b_2)$. If $b = \text{Min}\{b_1, b_2\}$ then $[x, b)$ is a member of $\mathscr{B}$ and

$$x \in [x, b) \subset [a_1, b_1) \cap [a_2, b_2).$$

It follows (see Note (iii)) that $\mathscr{B}$ satisfies condition (B2). Therefore $\mathscr{B}$ is a synthetic basis for $\mathbf{R}$.

Let $\mathscr{T}$ be the topology with $\mathscr{B}$ as basis. Then $\mathscr{T}$ is strictly finer than the usual topology on $\mathbf{R}$. To see this, recall that since the set of all open intervals is a basis for the usual topology, it is enough (by Note (ii)) to show that each open interval is a member of $\mathscr{T}$. Let $a, b \in \mathbf{R}$ with $a < b$. Let (a_n) be the strictly decreasing sequence of real numbers (that is, $a_{n+1} < a_n$ for all n in $\mathbf{N}$) defined by

$$a_n = a + \frac{b-a}{n+1}.$$

Then $[a_n, b) \subset (a, b)$ for each n, and so

$$\bigcup_{n=1}^{\infty} [a_n, b) \subset (a, b).$$

On the other hand, if $t \in (a, b)$, then, since $\lim_{n \to \infty} a_n = a$, there exists some positive integer m such that $a < a_m < t$ so that $t \in [a_m, b)$. Hence

$$(a, b) \subset \bigcup_{n=1}^{\infty} [a_n, b).$$

It follows that

$$(a, b) = \bigcup_{n=1}^{\infty} [a_n, b),$$

so that $(a, b) \in \mathscr{T}$, as required.

And as $[a, b)$ is not open in the usual topology but is open in $\mathscr{T}$, $\mathscr{T}$ is not equal to the usual topology.

SAQ 10 Let $\mathscr{T}$ be the topology for $\mathbf{R}$ introduced in Example 3, and let $\mathscr{B}$ be the subset of $\mathscr{T}$ consisting of all singleton subsets $\{x\}$ where $x \in \mathbf{R}$ and $x \neq 0$ together with all subsets $\mathbf{R} - F$ where F is finite and $0 \notin F$. Show that $\mathscr{B}$ is a basis for $\mathscr{T}$.

SAQ 11 Let $\mathscr{B}$ be a set of subsets of a non-empty set A with the properties: (i) $A \in \mathscr{B}$; (ii) if $B, C \in \mathscr{B}$ then either $B \subset C$ or $C \subset B$. Prove that $\mathscr{B}$ is a synthetic basis for A.

SAQ 12 (i) Show that the collection of all closed intervals $[a,b]$, where $a \leq b$, is a basis for the discrete topology on $\mathbf{R}$.

(ii) Show that the collection of all closed intervals $[a,b]$, where $a < b$, is not a synthetic basis for $\mathbf{R}$.

SOLUTION TO SAQ 10 If $V \in \mathscr{T}$ and $0 \notin V$ then writing $V = \bigcup\limits_{x \in V} \{x\}$ expresses V as a union of members of $\mathscr{B}$. If $V \in \mathscr{T}$ and $0 \in V$ then $\mathbf{R} - V$ is a finite subset, F say, of $\mathbf{R}$. Then $V = \mathbf{R} - F$, and since $0 \in V$ it follows that $0 \notin F$. Therefore $V \in \mathscr{B}$. Thus $\mathscr{B}$ is a basis for the topology $\mathscr{T}$. ■

SOLUTION TO SAQ 11 By (i), $\mathscr{B}$ satisfies the condition (B1). If $B, C \in \mathscr{B}$, then, by (ii), either $B \cap C = B$ or $B \cap C = C$; therefore $B \cap C \in \mathscr{B}$ and so condition (B2) is satisfied. Thus $\mathscr{B}$ is a synthetic basis for A. ■

SOLUTION TO SAQ 12 (i) Since this collection contains all singleton subsets $\{a\}$, $a \in \mathbf{R}$, it follows from Example 8 that it is a basis for the discrete topology on $\mathbf{R}$.

(ii) Since, for example, $[0,1] \cap [1,2] = \{1\}$ which is not a union of members of the collection, condition (B2) is not satisfied; so the collection is not a synthetic basis for $\mathbf{R}$. ■

4.4 Subspaces

READ S from the beginning of Section 3.4 on page 51 to page 52, line -10.

NOTES (i) *Page 51, line* -7

Let us check that $\{H, \mathscr{T}_H\}$ is a topological space. Since $\varnothing, A \in \mathscr{T}$, it follows that $H \cap \varnothing = \varnothing, H \cap A = H \in \mathscr{T}_H$; and so (T1) is satisfied. Also, (T2) holds for $\mathscr{T}_H$; for if U, $V \in \mathscr{T}$, then $U \cap V \in \mathscr{T}$ and

$$(U \cap H) \cap (V \cap H) = (U \cap V) \cap H \in \mathscr{T}_H.$$

Finally, let $(U_i)_{i \in I}$ be a family of members of $\mathscr{T}$. Then by the result of SAQ 9 of *Unit 1*,

$$\bigcup_{i \in I} (U_i \cap H) = (\bigcup_{i \in I} U_i) \cap H.$$

Since $\bigcup\limits_{i \in I} U_i \in \mathscr{T}$, we can see that (T3) is satisfied. Therefore $\mathscr{T}_H$ is a topology for H.

(ii) *Page 51, line* -3

One way of actually identifying the induced topology is to use bases; we now show how a basis for a topological space gives rise to a basis for a subspace of it.

Let $\mathscr{T}$ be a topology on a set A, let $H \subset A$ and let $\mathscr{T}_H$ be the induced topology on H. Also, let $\mathscr{B}$ be a basis for the topology $\mathscr{T}$ and let $\mathscr{B}_H = \{B \cap H : B \in \mathscr{B}\}$. Then $\mathscr{B}_H \subset \mathscr{T}_H$. Moreover, if $V \in \mathscr{T}_H$ then

$V = U \cap H$ where $U \in \mathscr{T}$, so that $U = \bigcup\limits_{i \in I} B_i$, where $B_i \in \mathscr{B}$ for each i. Therefore

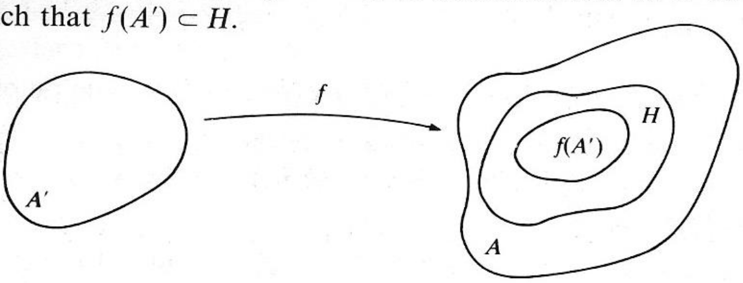
$$V = (\bigcup_{i \in I} B_i) \cap H = \bigcup_{i \in I} (B_i \cap H).$$

Thus each member of $\mathscr{T}_H$ is a union of members of $\mathscr{B}_H$. Hence $\mathscr{B}_H$ is a basis for $\mathscr{T}_H$.

(iii) *Page 52, Proposition 3.4.2(a)*

An important step in the proof of this result establishes that if $\{A, \mathscr{T}\}$ is a topological space and $H \subset A$ then the inclusion map $i : H \to A$ is $(\mathscr{T}_H, \mathscr{T})$-continuous. In fact, $\mathscr{T}_H$ is the coarsest topology $\mathscr{N}$ on H such that i is $(\mathscr{N}, \mathscr{T})$-continuous. For let $\mathscr{N}$ be a topology on H such that i is $(\mathscr{N}, \mathscr{T})$-continuous. If $U \in \mathscr{T}$ then $i^{-1}(U) = U \cap H \in \mathscr{N}$. Hence $\mathscr{T}_H \subset \mathscr{N}$.

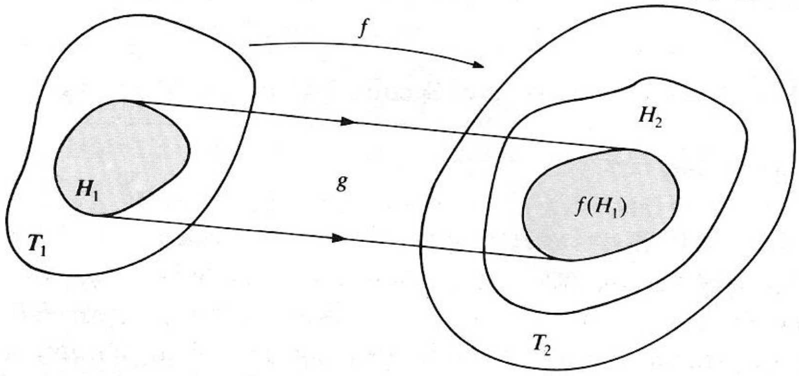
(iv) *Page 52, Proposition 3.4.2(b)*
A reformulated version of this result is often used. Let $\{A', \mathscr{T}'\}$ and $\{A, \mathscr{T}\}$ be topological spaces, let $f: A' \to A$ be a function and let H be a subset of A such that $f(A') \subset H$.



Then we can define $g: A' \to H$ by putting $g(x) = f(x)$ for each x in A'. If $i: H \to A$ is the inclusion map defined in Section 1.3 of *Unit 1*, then $f = i \circ g$.

Hence Proposition 3.4.2(b) states that g is continuous if and only if f is continuous.

(v) *Page 52, Proposition 3.4.2*
The two parts of Proposition 3.4.2 can be combined to give the following important result. If T_1, T_2 are topological spaces, $f: T_1 \to T_2$ is continuous and H_1, H_2 are subspaces of T_1, T_2 respectively such that $f(H_1) \subset H_2$, then



$g: H_1 \to H_2$ given by $g(x) = f(x)$, $x \in H_1$, is continuous. For, by part (a), $f|H_1: H_1 \to T_2$ is continuous, and by applying part (b) to this function we see that g is continuous. ▲

EXAMPLE 10 Let **R** have its usual topology and let us examine some subspaces of **R**.

(a) Since the set of all open intervals is a basis for the usual topology on **R**, it follows that a basis for the induced topology on $[0,1]$ consists of all sets of the form $[0,1] \cap (a,b)$ where $a, b \in \mathbf{R}$ and $a < b$. Thus a basis for the topology of the subspace $[0,1]$ consists of all sets of the following three forms:

(i) $[0,b)$ where $0 < b \leq 1$;

(ii) (a,b) where $0 \leq a < b \leq 1$;

(iii) $(a,1]$ where $0 \leq a < 1$.

(b) Let $B = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and let $\mathscr{V}$ consist of all subsets V of B such that either $0 \notin V$ or $B - V$ is finite. Let $\mathscr{T}$ denote the topology on B induced by the usual topology on **R**. We shall show that $\mathscr{V} = \mathscr{T}$.

Let $V \in \mathscr{T}$; then there exists an open set W of **R** such that $V = W \cap B$.

If $0 \notin W$ then $V = W \cap B \in \mathscr{V}$. If $0 \in W$ then there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset W$. But there exists a positive integer N such that $0 < \dfrac{1}{n} < \varepsilon$ for all $n > N$. Thus

$$B - (W \cap B) \subset \left\{1, \frac{1}{2}, \ldots, \frac{1}{N}\right\},$$

and so $V = W \cap B \in \mathscr{V}$. Thus $\mathscr{T} \subset \mathscr{V}$.

To prove the reverse inclusion, we first show that the singleton $\left\{\frac{1}{n}\right\}$ is in $\mathscr{T}$ for each $n \in \mathbf{N}$. Now $(\frac{1}{2}, 2) \cap B = \{1\}$ and $\left(\frac{1}{n+1}, \frac{1}{n-1}\right) \cap B = \left\{\frac{1}{n}\right\}$ for each integer $n \geq 2$, so $\left\{\frac{1}{n}\right\}$ is in $\mathscr{T}$ for each $n \in \mathbf{N}$.

Now let $V \in \mathscr{V}$. If $0 \notin V$ then for some subset M of $\mathbf{N}$, $V = \bigcup_{n \in M} \left\{\frac{1}{n}\right\}$, so that $V \in \mathscr{T}$. If $0 \in V$, then $B - V$ is finite. Hence there exists N in $\mathbf{N}$ such that $\frac{1}{n} \in V$ if $n > N$. So $\left\{x \in V : x < \frac{1}{N}\right\}$ is in $\mathscr{T}$ since

$$\left\{x \in V : x < \frac{1}{N}\right\} = \left(-1, \frac{1}{N}\right) \cap B.$$

Also, $\left\{x \in V : x \geq \frac{1}{N}\right\}$ is the union of the singleton sets $\left\{\frac{1}{n}\right\}$ for those integers n such that $n \leq N$ and $\frac{1}{n} \in V$. Hence $\left\{x \in V : x \geq \frac{1}{N}\right\}$ is in $\mathscr{T}$. Since

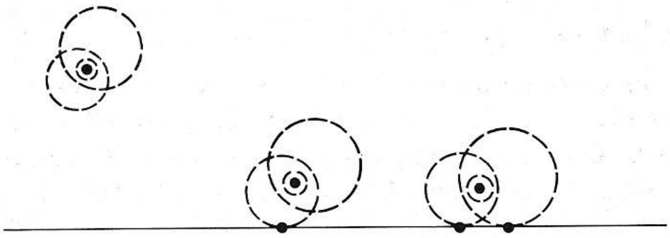
$$V = \left\{x \in V : x < \frac{1}{N}\right\} \cup \left\{x \in V : x \geq \frac{1}{N}\right\},$$

it follows that $V \in \mathscr{T}$. Hence if $V \in \mathscr{V}$ then $V \in \mathscr{T}$, i.e. $\mathscr{V} \subset \mathscr{T}$, so that $\mathscr{V} = \mathscr{T}$.

EXAMPLE 11 Let

$$A = \{(x_1, x_2) \in \mathbf{R}^2 : x_2 \geq 0\}$$

be the upper half-plane. We define a synthetic basis $\mathscr{B}$ for A as follows: $B \in \mathscr{B}$ if and only if either (i) B is the interior of a disc lying in the upper half-plane or (ii) B is the interior of a disc lying in the upper half-plane, tangent to the line $x_2 = 0$, together with the point of tangency. The following diagrams should help you to see that $\mathscr{B}$ satisfies conditions (B1) and (B2).



Let $\mathscr{T}$ be the topology for A which has $\mathscr{B}$ as basis and let $T = \{A, \mathscr{T}\}$. If

$$H_1 = \{(x_1, x_2) \in A : x_2 > 0\},$$

then the topology of H_1 as a subspace of T coincides with its topology as a subspace of $\mathbf{R}^2$ (with the usual topology). But if

$$H_2 = \{(x_1, x_2) \in A : x_2 = 0\},$$

then the subspace H_2 of T is discrete since each point of H_2 is the intersection of H_2 with an open set of $\mathscr{T}$.

SAQ 13 Consider the following topology $\mathscr{T}$ on $A = \{a, b, c, d, e\}$:

$$\mathscr{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, e\}, \{a, b, c, d\}, A\}.$$

Find the induced topology on $H = \{a, b, c\}$.

SAQ 14 Let $\mathbf{R}$ have the topology $\mathscr{T}$. Find the induced topology on the subset $\mathbf{Z}$ of $\mathbf{R}$ in the following cases:

(i) $\mathscr{T}$ is the usual topology;

(ii) $\mathscr{T}$ is the topology introduced in Example 3 (on page 96);

(iii) $\mathscr{T}$ is the topology introduced in Example 9 (on page 104).

SAQ 15 Let H be any open set of a topological space T and let V be any open set of the subspace H of T.

(i) Show that V is an open set of T.

(ii) Give an example of a subspace H of a topological space T and an open subset V of H such that V is *not* an open set of T.

SAQ 16 Let $\mathscr{T}$ be a topology for a set A, let H and K be subsets of A such that $H \subset K$, and let $\mathscr{T}_H$, $\mathscr{T}_K$ be the topologies induced by $\mathscr{T}$ on H, K respectively.

Prove that $\mathscr{T}_H$ coincides with the topology on H induced by $\mathscr{T}_K$.

SAQ 17 Let $\{A,d\}$ be a metric space, let $\mathscr{T}$ be the topology on A arising from d and let H be a subset of A. Show that the topology $\mathscr{T}_H$ on H induced by $\mathscr{T}$ is the same as the topology arising from the metric d_H on H induced by d.

[Hint: Let $\mathscr{U}$ be the topology on H arising from d_H; using that $\{B_\varepsilon(x;d_H):x\in H\}$ is a basis for $\mathscr{U}$, show that $\mathscr{U} \subset \mathscr{T}_H$. Conversely, show that each set $V\in\mathscr{T}_H$ is d_H-open.]

SOLUTION TO SAQ 13 The induced topology on H is, from Definition 3.4.1,

$$\mathscr{T}_H = \{\varnothing, \{a\}, \{a,b\}, \{a,c\}, H\}. \quad \blacksquare$$

SOLUTION TO SAQ 14

(i) If $n\in\mathbf{Z}$ then $(n-\frac{1}{2}, n+\frac{1}{2})\in\mathscr{T}$. But $\mathbf{Z}\cap(n-\frac{1}{2},n+\frac{1}{2}) = \{n\}$. Hence $\{n\}$ is an open set of the subspace $\mathbf{Z}$ for each n in $\mathbf{Z}$. Thus the induced topology on $\mathbf{Z}$ is the discrete topology.

(ii) Let $\mathscr{U}$ be the induced topology on $\mathbf{Z}$ and let $\mathscr{V}$ consist of those subsets V of $\mathbf{Z}$ such that either $0\notin V$ or $\mathbf{Z}-V$ is finite. If $V\subset\mathbf{Z}$ and $0\notin V$ then $V\in\mathscr{T}$, so that $V\cap\mathbf{Z}=V\in\mathscr{U}$. If $V\subset\mathbf{Z}$ and $\mathbf{Z}-V=F$ where F is finite, then $V=\mathbf{Z}-F=\mathbf{Z}\cap(\mathbf{R}-F)$ and $\mathbf{R}-F\in\mathscr{T}$, since its complement F is finite, and hence $V\in\mathscr{U}$. Thus $\mathscr{V}\subset\mathscr{U}$. Conversely, if $U\in\mathscr{U}$ then $U=\mathbf{Z}\cap W$ where $W\in\mathscr{T}$. Hence either $0\notin W$ or $\mathbf{R}-W$ is finite. If $0\notin W$ then $0\notin U$. If $\mathbf{R}-W$ is finite then

$$\mathbf{Z}-U=\mathbf{Z}-(\mathbf{Z}\cap W)=\mathbf{Z}\cap(\mathbf{R}-W),$$

so that $\mathbf{Z}-U$ is finite. Thus $U\in\mathscr{V}$. Therefore $\mathscr{U}\subset\mathscr{V}$ and so $\mathscr{U}=\mathscr{V}$.

(iii) If $n\in\mathbf{Z}$ then $[n,n+1)\in\mathscr{T}$. But $\mathbf{Z}\cap[n,n+1)=\{n\}$. Thus the induced topology on $\mathbf{Z}$ is the discrete topology. $\blacksquare$

Remark. Notice that the two different (but comparable) topologies on $\mathbf{R}$ in (i) and (iii) determine the same induced topology on $\mathbf{Z}$.

SOLUTION TO SAQ 15

(i) Since V is an open set of the subspace H, there exists W open in T such that $V=W\cap H$. But H is open in T so that V is the intersection of two sets open in T from which it follows that V is open in T.

(ii) Let T be $\mathbf{R}$ with the usual topology, $H=(0,1]$ and $V=(\frac{1}{2},1]$. Then V is an open subset of the subspace H of $\mathbf{R}$, but it is not open in $\mathbf{R}$. $\blacksquare$

SOLUTION TO SAQ 16 Let $\mathscr{U}$ be the topology on H induced by the topology $\mathscr{T}_K$ on K. If $U\in\mathscr{U}$ then $U=V\cap H$ where $V\in\mathscr{T}_K$. But since $V\in\mathscr{T}_K$, we know that $V=W\cap K$ where $W\in\mathscr{T}$. Therefore

$$U=(W\cap K)\cap H=W\cap H$$

since $H\subset K$. Thus $U\in\mathscr{T}_H$.

Conversely, if $U\in\mathscr{T}_H$ then $U=W\cap H$ where $W\in\mathscr{T}$. But $W\cap H=(W\cap K)\cap H$ and $W\cap K\in\mathscr{T}_K$, from which we can see that $U\in\mathscr{U}$. Hence $\mathscr{U}=\mathscr{T}_H$. $\blacksquare$

SOLUTION TO SAQ 17 First we notice that if $x \in H$ and $\varepsilon > 0$ then

$$\begin{aligned} B_\varepsilon(x; d_H) &= \{y \in H : d_H(x, y) < \varepsilon\} \\ &= \{y \in H : d(x, y) < \varepsilon\} \\ &= H \cap B_\varepsilon(x; d). \end{aligned}$$

Thus each set $B_\varepsilon(x; d_H)$ is a member of $\mathscr{T}_H$. Since such sets form a basis for the topology $\mathscr{U}$ on H induced by d_H, we see that $\mathscr{U} \subset \mathscr{T}_H$.

Conversely, if $V \in \mathscr{T}_H$, then there exists $W \in \mathscr{T}$ such that $V = W \cap H$. If $x \in V$, then $x \in W$ so there exists $\varepsilon(x) > 0$ such that $B_{\varepsilon(x)}(x; d) \subset W$. Then

$$B_{\varepsilon(x)}(x; d_H) = B_{\varepsilon(x)}(x; d) \cap H \subset W \cap H \subset V.$$

Therefore V is d_H-open, i.e. $V \in \mathscr{U}$. Hence $\mathscr{U} = \mathscr{T}_H$. ■

4.5 Topological Products

Given two topological spaces $T_1 = \{A_1, \mathscr{T}_1\}$ and $T_2 = \{A_2, \mathscr{T}_2\}$, we can define an associated topology on the Cartesian product $A_1 \times A_2$. Then $A_1 \times A_2$ with this topology is called the *Cartesian product* of T_1 and T_2. You might find it helpful to look again at *Unit 1* Section 1.4 on Cartesian products, before studying this section.

READ S all of Section 3.5 from page 53, with the exception of the starred passages.

NOTES

(i) *Page 54, line −10*
The proof of this formula is similar to that of SAQ 21(b) of *Unit 1*.

(ii) *Page 55, Proposition 3.5.2*
This result is simply a restatement of the fact that the collection of sets of the form $U_1 \times U_2$, where U_1, U_2 are open in T_1, T_2 respectively, is a basis for the topology of $T_1 \times T_2$. An extension of this result is frequently used; namely that if $\mathscr{B}_1$ and $\mathscr{B}_2$ are bases for the topologies of T_1 and T_2 respectively, then a basis for the topology of $T_1 \times T_2$ consists of all sets of the form $B_1 \times B_2$ where $B_1 \in \mathscr{B}_1$, $B_2 \in \mathscr{B}_2$.

To see this, let W be an open set of $T_1 \times T_2$ and let $(x, y) \in W$. By Proposition 3.5.2, there exist U_1, U_2 open in T_1, T_2 respectively such that $(x, y) \in U_1 \times U_2 \subset W$. Since $x \in U_1$ and $\mathscr{B}_1$ is a basis for the topology of T_1, there exists $B_1 \in \mathscr{B}_1$ such that $x \in B_1 \subset U_1$. Similarly, there exists $B_2 \in \mathscr{B}_2$ such that $y \in B_2 \subset U_2$. Then $(x, y) \in B_1 \times B_2 \subset U_1 \times U_2 \subset W$.

(iii) *Page 56, lines 2–5*
This formula was established in SAQ 24 of *Unit 1*. ▲

EXAMPLE 12 Let $T_1 = \{A, \mathscr{T}\}$ where $A = \{a, c\}$ and $\mathscr{T} = \{\varnothing, \{a\}, A\}$ and let $T_2 = \{B, \mathscr{U}\}$ where $B = \{b, d\}$ and $\mathscr{U} = \{\varnothing, \{b\}, B\}$. Let us determine the topological product $T_1 \times T_2$.

Its underlying set is

$$A \times B = \{\alpha, \beta, \gamma, \delta\}$$

where

$$\alpha = (a, b),\ \beta = (a, d),\ \gamma = (c, b),\ \delta = (c, d).$$

A basis $\mathscr{B}$ for the topology of $T_1 \times T_2$ consists of the sets $\{a\} \times \{b\}, \{a\} \times B, A \times \{b\}$ and $A \times B$. Thus

$$\mathscr{B} = \{\{\alpha\}, \{\alpha, \beta\}, \{\alpha, \gamma\}, A \times B\}.$$

Therefore the topology of $T_1 \times T_2$ is

$$\{\varnothing, \{\alpha\}, \{\alpha, \beta\}, \{\alpha, \gamma\}, \{\alpha, \beta, \gamma\}, A \times B\}.$$

EXAMPLE 13 Let $M_1 = \{A_1, d_1\}$ and $M_2 = \{A_2, d_2\}$ be metric spaces and, for $i = 1, 2$, let $T_i = \{A_i, \mathscr{T}_i\}$, where $\mathscr{T}_i$ is the topology for A_i consisting of the d_i-open sets. We show that the product topology $\mathscr{T}$ for $A_1 \times A_2$ consists of the d-open sets where d is the metric on $A_1 \times A_2$ defined as follows: if $x = (x_1, x_2) \in A_1 \times A_2$ and $y = (y_1, y_2) \in A_1 \times A_2$ then

$$d(x, y) = \text{Max}\,\{d_1(x_1, y_1),\, d_2(x_2, y_2)\}.$$

(*Remark.* In Example 2.2.7, **S** page 26, we met several different metrics on a Cartesian product $A_1 \times A_2$; and in Section 2.4 of **S** saw that in a sense they were all equivalent. Here we are using one particular metric on $A_1 \times A_2$ to specify the product topology, but of course many other metrics on $A_1 \times A_2$ would give rise to the same product topology.)

Let $\mathscr{U}$ denote the set of d-open sets of $A_1 \times A_2$. A basis for the metric topology $\mathscr{U}$ consists of all sets of the form $B_\varepsilon(a; d)$ where $a = (a_1, a_2) \in A_1 \times A_2$ and $\varepsilon > 0$. Since

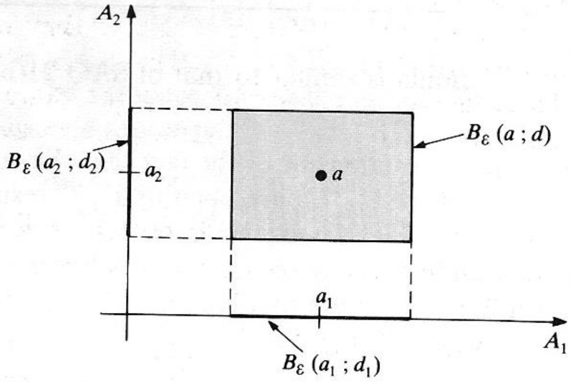
$$B_\varepsilon(a; d) = B_\varepsilon(a_1; d_1) \times B_\varepsilon(a_2; d_2)$$

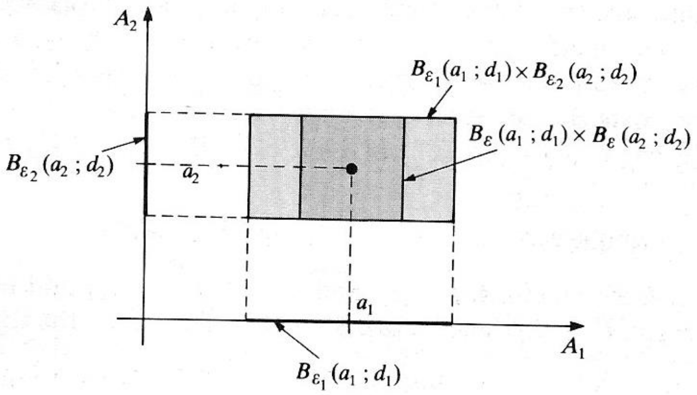
and $B_\varepsilon(a_i; d_i) \in \mathscr{T}_i$ for $i = 1$, 2, we see that $B_\varepsilon(a; d) \in \mathscr{T}$. It follows that $\mathscr{U} \subset \mathscr{T}$.

Also, a basis for $\mathscr{T}$ consists of the sets $U_1 \times U_2$ where $U_i \in \mathscr{T}_i$ for $i = 1, 2$. If $a = (a_1, a_2) \in U_1 \times U_2$, then $a_i \in U_i$ for $i = 1, 2$, so that there exists $\varepsilon_i > 0$ such that $B_{\varepsilon_i}(a_i; d_i) \subset U_i$. Let $\varepsilon = \text{Min}\,\{\varepsilon_1, \varepsilon_2\}$. Then

$$\begin{aligned} B_\varepsilon(a; d) &= B_\varepsilon(a_1; d_1) \times B_\varepsilon(a_2; d_2) \\ &\subset B_{\varepsilon_1}(a_1; d_1) \times B_{\varepsilon_2}(a_2; d_2) \subset U_1 \times U_2. \end{aligned}$$

Hence $U_1 \times U_2 \in \mathscr{U}$. It follows that $\mathscr{T} \subset \mathscr{U}$. Therefore $\mathscr{T} = \mathscr{U}$.




A_2
$B_{\varepsilon_1}(a_1; d_1) \times B_{\varepsilon_2}(a_2; d_2)$
$B_\varepsilon(a_1; d_1) \times B_\varepsilon(a_2; d_2)$
$B_{\varepsilon_2}(a_2; d_2)$
a_2
a_1
A_1
$B_{\varepsilon_1}(a_1; d_1)$

Now play the last part of the tape for Unit 4

SAQ 18 Let $\{A_1, \mathscr{T}_1\}$ and $\{A_2, \mathscr{T}_2\}$ be topological spaces. Determine the product topology for $A_1 \times A_2$ in the following cases.

(i) $\mathscr{T}_1$ and $\mathscr{T}_2$ are both indiscrete topologies.

(ii) $\mathscr{T}_1$ is the discrete topology and $\mathscr{T}_2$ is the indiscrete topology.

(iii) $\mathscr{T}_1$ and $\mathscr{T}_2$ are both discrete topologies.

SAQ 19 Let $p_1 : T_1 \times T_2 \to T_1$ be the first projection of a topological product and let V be an open set of $T_1 \times T_2$. Show that $p_1(V)$ is an open set of T_1.

SAQ 20 Let $T = \{A, \mathscr{T}\}$ be a topological space and let

$$\Delta = \{(x, y) \in A \times A : x = y\}.$$

Show that if Δ is an open set of the topological product $T \times T$ then T is a discrete space.

***SAQ 21** For $i = 1, 2$, let S_i and T_i be topological spaces and let $f_i : S_i \to T_i$ be continuous. Show that

$$f_1 \times f_2 : S_1 \times S_2 \to T_1 \times T_2$$

is continuous.

(The definition of $f_1 \times f_2$ was given in Section 1.4 of *Unit 1*.)

SOLUTION TO SAQ 18

(i) The product topology is the indiscrete topology.

(ii) The product topology consists of all sets $B \times A_2$ where $B \in 2^{A_1}$. (This includes $\varnothing$, of course.)

(iii) A basis for the product topology consists of all sets

$$\{a_1\} \times \{a_2\} = \{(a_1, a_2)\}, \qquad (a_1, a_2) \in A_1 \times A_2,$$

so that the product topology is the discrete topology on $A_1 \times A_2$. ■

SOLUTION TO SAQ 19 Let $x \in p_1(V)$. Then there exists y in T_2 such that $(x, y) \in V$. It follows from **S** Proposition 3.5.2 that there exist U_1, U_2 open in T_1, T_2 respectively such that $(x, y) \in U_1 \times U_2 \subset V$. Then $x \in p_1(U_1 \times U_2) \subset p_1(V)$. But $p_1(U_1 \times U_2) = U_1$, which is an open set of T_1. Therefore (by SAQ 4) $p_1(V)$ is open in T_1. ■

SOLUTION TO SAQ 20 Suppose that Δ is an open set and let $a \in A$. Then $(a, a) \in \Delta$. It follows from **S** Proposition 3.5.2 that there exist V, W open in T such that $(a, a) \in V \times W \subset \Delta$. If $a' \in V$ where $a' \neq a$, then $(a', a) \in V \times W \subset \Delta$ where $a' \neq a$, which is a contradiction of the definition of Δ. Therefore $V = \{a\}$. Since each singleton subset of T is an open set, it follows that T is a discrete space. ■

SOLUTION TO SAQ 21 For $i = 1, 2$ let

$$p_i : S_1 \times S_2 \to S_i \quad \text{and} \quad q_i : T_1 \times T_2 \to T_i$$

denote the projections of the topological products. It was established in Section 1.4 of *Unit 1* that

$$q_1 \circ (f_1 \times f_2) = f_1 \circ p_1 \quad \text{and} \quad q_2 \circ (f_1 \times f_2) = f_2 \circ p_2.$$

By **S** Propositions 3.5.3 and 3.1.8, the maps $f_1 \circ p_1$ and $f_2 \circ p_2$ are continuous. Therefore $f_1 \times f_2$ is a map into $T_1 \times T_2$ such that $q_1 \circ (f_1 \times f_2)$ and $q_2 \circ (f_1 \times f_2)$ are continuous. It follows from **S** Proposition 3.5.4 that $f_1 \times f_2$ is continuous. ■

4.6 Objectives of Unit 4

After working through this unit, you should know:

(1) the definitions of a topology for a set, of a topological space and of the open sets of a topological space;

(2) the definition of continuity of a function between topological spaces and the main properties of continuous functions;

(3) the definition of a basis for a topology and of a synthetic basis for a set, and the relation between the two ideas;

(4) how a topology on a set can be specified in terms of a basis;

(5) the definitions of the induced topology on a subset of a topological space and of a subspace of a topological space;

(6) the definition of the product of topological spaces and the main properties of the topological product.

4.7 Revision Questions

SAQ 22 Let $\mathscr{T}$ be the set of subsets of $\mathbf{R}$ consisting of $\varnothing$, $\mathbf{R}$ and all unbounded open intervals of the form $(-\infty, a)$, $a \in \mathbf{R}$. Show that $\mathscr{T}$ is a topology on $\mathbf{R}$ which is strictly coarser than the usual topology on $\mathbf{R}$. (This topology is called the *left ray topology*.)

SAQ 23 Let $f: \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = kx, \quad x \in \mathbf{R},$$

where $k \in \mathbf{R}$. Let $\mathscr{T}$ be the topology on $\mathbf{R}$ introduced in SAQ 22 and let $\mathscr{U}$ be the usual topology on $\mathbf{R}$. For what values of the real number k is f
(i) $(\mathscr{T}, \mathscr{T})$-continuous, (ii) $(\mathscr{U}, \mathscr{T})$-continuous, (iii) $(\mathscr{T}, \mathscr{U})$-continuous?

SAQ 24 Let $\mathbf{R}$ have its usual topology and let B be the subspace $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ of $\mathbf{R}$. Prove that $f: B \to \mathbf{R}$ is continuous if and only if

$$f(0) = \lim_{n \to \infty} f\left(\frac{1}{n}\right).$$

[Hint: Refer back to Example 10(b).]

SAQ 25 (i) Let $\mathbf{R}$ have its usual topology and let $\sigma: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by

$$\sigma(x_1, x_2) = x_1 + x_2, \quad (x_1, x_2) \in \mathbf{R} \times \mathbf{R}.$$

Prove that σ is continuous.

[Hint: Consider projections.]

(ii) Let f, g be continuous real-valued functions on a topological space T. Prove that $f + g$ is a continuous real-valued function on T.

SAQ 26 Let $\mathscr{T}$ denote the topology for $\mathbf{R}$ defined in Example 9 and let $T = \{\mathbf{R}, \mathscr{T}\}$. Show that (i) the subspace

$$H = \{(x_1, x_2) \in \mathbf{R} \times \mathbf{R} : x_1 + x_2 = 0\}$$

of $T \times T$ is discrete, but that (ii) the subspace

$$K = \{(x_1, x_2) \in \mathbf{R} \times \mathbf{R} : x_1 = x_2\}$$

of $T \times T$ is not discrete.

SOLUTION TO SAQ 22 Condition (T1) is satisfied by the definition of $\mathscr{T}$. Let us put $V_a = (-\infty, a)$ for each a in $\mathbf{R}$. If $a, b \in \mathbf{R}$ and $a < b$ then $V_a \subset V_b$ so that $V_a \cap V_b = V_a$. Thus (T2) is satisfied.

To see that (T3) is satisfied, let S be a non-empty subset of $\mathbf{R}$ and let $V = \bigcup_{a \in S} V_a$. We must show that $V \in \mathscr{T}$. If S is not bounded above, then, for each x in $\mathbf{R}$ there exists a in S such that $x < a$, from which it follows that $x \in V_a \subset V$. Therefore $V = \mathbf{R} \in \mathscr{T}$. If S is bounded above, let $b = \sup S$. Then $a \leq b$ for each a in S so that $V_a \subset V_b$ for each a in S and hence $V \subset V_b$. But if $x \in V_b$ then $x < b$ so that there exists a in S such that $x < a$, from which it follows that $x \in V_a \subset V$. Therefore $V = V_b \in \mathscr{T}$, and so (T3) is satisfied. Each member of $\mathscr{T}$ is open in $\mathbf{R}$ with its usual topology, but there are open sets of $\mathbf{R}$ which are not members of $\mathscr{T}$, for example any bounded open interval (a, b). Hence $\mathscr{T}$ is a topology on $\mathbf{R}$ which is strictly coarser than the usual topology. ■

SOLUTION TO SAQ 23 (i) If $k > 0$, then for each a in $\mathbf{R}$,

$$f^{-1}((-\infty, a)) = \{x \in \mathbf{R} : kx < a\} = \left\{x \in \mathbf{R} : x < \frac{a}{k}\right\}$$

$$= \left(-\infty, \frac{a}{k}\right) \in \mathscr{T}.$$

Thus f is $(\mathscr{T}, \mathscr{T})$-continuous if $k > 0$.

If $k = 0$, then f is a constant mapping and so is $(\mathscr{T}, \mathscr{T})$-continuous by **S** Proposition 3.1.10(b).

If $k < 0$, then for each a in $\mathbf{R}$,

$$f^{-1}((-\infty, a)) = \{x \in \mathbf{R} : kx < a\} = \left\{x \in \mathbf{R} : x > \frac{a}{k}\right\},$$

so that $f^{-1}((-\infty, a)) \notin \mathscr{T}$. Thus f is not $(\mathscr{T}, \mathscr{T})$-continuous if $k < 0$.

(ii) If $k = 0$, then f is a constant mapping and so f is continuous by Proposition 3.1.10(b).

If $k \neq 0$, then for each a in $\mathbf{R}$,

$$f^{-1}((-\infty, a)) = \begin{cases} \left(-\infty, \dfrac{a}{k}\right), & \text{if } k > 0, \\ \left(\dfrac{a}{k}, \infty\right), & \text{if } k < 0. \end{cases}$$

Since $f^{-1}((-\infty, a)) \in \mathscr{U}$ for every a, we see that f is $(\mathscr{U}, \mathscr{T})$-continuous. Therefore f is $(\mathscr{U}, \mathscr{T})$-continuous for all values of k.

(iii) If $k = 0$ then f is a constant mapping and so is $(\mathscr{T}, \mathscr{U})$-continuous. If $k \neq 0$ then

$$f^{-1}((0,1)) = \{x \in \mathbf{R} : 0 < kx < 1\} = \begin{cases} \left(0, \dfrac{1}{k}\right), & \text{if } k > 0, \\ \left(\dfrac{1}{k}, 0\right), & \text{if } k < 0. \end{cases}$$

Thus $(0,1) \in \mathscr{U}$ and $f^{-1}((0,1)) \notin \mathscr{T}$ if $k \neq 0$, so that f is not $(\mathscr{T}, \mathscr{U})$-continuous. Therefore f is $(\mathscr{T}, \mathscr{U})$-continuous if and only if $k = 0$. ■

SOLUTION TO SAQ 24 (Recall that the topology of the subspace B of $\mathbf{R}$ was described in Example 10.)

Suppose first that f is a continuous function and let us prove that $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0)$. Let $\varepsilon > 0$ be given.

If $V = (f(0) - \varepsilon, f(0) + \varepsilon)$ then V is an open set of $\mathbf{R}$, so that $f^{-1}(V)$ is an open set of B. Since $0 \in f^{-1}(V)$, it follows that $B - f^{-1}(V)$ is finite. Therefore there exists a positive integer N such that

$$B - f^{-1}(V) \subset \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N}\right\}.$$

It follows that if $n \in \mathbf{N}$ and $n > N$ then $\frac{1}{n} \in f^{-1}(V)$, so that

$f(0) - \varepsilon < f\left(\frac{1}{n}\right) < f(0) + \varepsilon$. Hence

$$\left| f(0) - f\left(\frac{1}{n}\right) \right| < \varepsilon \quad \text{for all } n > N;$$

that is, $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0)$.

Conversely, suppose that $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0)$, and let us prove that $f : B \to \mathbf{R}$ is continuous. Let V be an open set of $\mathbf{R}$. If $0 \notin f^{-1}(V)$ then $f^{-1}(V)$ is an open set of B. If $0 \in f^{-1}(V)$ then $f(0) \in V$ so there exists $\varepsilon > 0$ such that $(f(0) - \varepsilon, f(0) + \varepsilon) \subset V$. Since $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0)$, there exists a positive integer N such that $\left| f\left(\frac{1}{n}\right) - f(0) \right| < \varepsilon$ for all $n > N$. Thus if $n \in \mathbf{N}$ and $n > N$ then $f\left(\frac{1}{n}\right) \in V$ so that $\frac{1}{n} \in f^{-1}(V)$. Therefore $B - f^{-1}(V)$ is a finite subset of B and so $f^{-1}(V)$ is an open set of B. Hence f is continuous. ■

SOLUTION TO SAQ 25

(i) By **S** Proposition 3.5.3, the projections $p_1, p_2: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are continuous. By the result of SAQ 25 of *Unit 3*, the real-valued function $p_1 + p_2$ is continuous. But if $x = (x_1, x_2) \in \mathbf{R} \times \mathbf{R}$ then

$$(p_1 + p_2)(x) = p_1(x) + p_2(x) = x_1 + x_2,$$

so that $\sigma = p_1 + p_2$ and σ is therefore continuous.

(ii) By **S** Proposition 3.5.4, $h: T \to \mathbf{R} \times \mathbf{R}$ given by

$$h(x) = (f(x), g(x)), \quad x \in T,$$

is continuous. It follows from **S** Proposition 3.1.8 that $\sigma \circ h: T \to \mathbf{R}$ is continuous. But if $x \in T$,

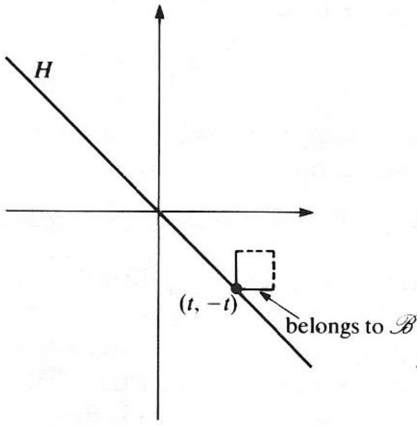
$$(\sigma \circ h)(x) = \sigma(f(x), g(x)) = f(x) + g(x) = (f + g)(x),$$

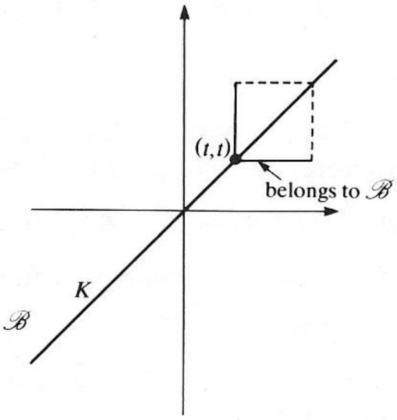
so that $f + g = \sigma \circ h$ and $f + g$ is thus continuous. ■

SOLUTION TO SAQ 26

A basis $\mathscr{B}$ for the topology of $T \times T$ consists of all sets of the form $[a, b) \times [c, d)$.

(i) Let $t \in \mathbf{R}$ so that $(t, -t) \in H$. Since $[t, t + 1) \times [-t, -t + 1) \in \mathscr{B}$ and $H \cap ([t, t + 1) \times [-t, -t + 1)) = \{(t, -t)\}$, we see that the induced topology on H is the discrete topology.





In particular, $\{(0, 0)\}$ is an open set of H.

(ii) Let us show that the subspace topology on K is not discrete, by proving that $\{(0, 0)\}$ is not an open set of K. If $B \in \mathscr{B}$ and $(0, 0) \in B$ then $B = [a, b) \times [c, d)$ where $a \leq 0 < b$ and $c \leq 0 < d$, so that Min $\{b, d\} > 0$. If $0 < t <$ Min $\{b, d\}$ then $(t, t) \in B$ and $(t, t) \neq (0, 0)$. Thus $\{(0, 0)\}$ is not an open set of K. ■

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